On choice of the regularization parameter in ill-posed problems with rough estimate of the noise level of the data

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Let $A\colon X \to Y$ be a linear bounded operator between real Hilbert spaces. We are interested in finding the minimum norm solution x_* of the ill-posed problem

$$Ax = y_*, \qquad y_* \in \mathcal{R}(A) \neq \overline{\mathcal{R}(A)}. \tag{1}$$

Instead of exact data y_*, noisy data y are given. We consider regularization of the problem (1) by Tikhonov method $x_\alpha = (\alpha I + A^*A)^{-1}A^*y$, in the case $X = Y$, $A = A^* \geq 0$ also by Lavrentiev method $x_\alpha = (\alpha I + A)^{-1}y$. The choice of the regularization parameter α depends on the knowledge about the noise level. Typically it is assumed that the exact noise level $\delta \geq \|y - y_*\|$ is given or no information about $\|y - y_*\|$ is available. The classical rules for the choice of the regularization parameter (e.g. the (modified) discrepancy principle) need the exact noise level: they fail in the case of underestimated noise level and give large error in the case of overestimated noise level. On the other hand, for heuristic rules that do not use the noise level (L-curve, GCV-rule etc), the convergence of the approximate solutions as $\|y - y_*\| \to 0$ can not be guaranteed (Bakushinskii's veto). Therefore we consider the case where the approximate noise level δ is given, but it is not known whether the inequality $\|y - y_*\| \leq \delta$ holds or not. For example, it may be known that with high probability $\delta / \|y - y_*\| \in [1/10, 10]$.

We formulate the following family of rules. Denote $B_\alpha = \sqrt{\alpha}(\alpha I + AA^*)^{-1/2}$, $D_\alpha = \alpha^{-1}AA^*B_\alpha^2$, $q_0 = 3/2$, $q_1 = 2q - 2$, $q_2 = 2q$, $t = 2$ for Tikhonov method, $B_\alpha = \alpha(\alpha I + A)^{-1}$, $D_\alpha = \alpha^{-1}AB_\alpha$, $q_0 = 4/3$, $q_1 = 3q/2 - 1$, $q_2 = 3q$, $t = 1$ for Lavrentiev method. Fix q, l, k such that q_2, $2k$, $2l \in N$, $l \geq 0$, $k \geq l/q$ and $q_0 \leq q < \infty$. Choose α as the largest solution of the equation

$$\kappa(\alpha)\|D_\alpha^k B_\alpha (Ax_\alpha - y)\|^{q/(q-1)} / \|D_\alpha^l B_\alpha^{q_1}(Ax_\alpha - y)\|^{1/(q-1)} = b\delta \tag{2}$$

where $b = \text{const}$; $\kappa(\alpha) = 1$, if $k = l/q$, and $\kappa(\alpha) = (1 + \alpha\|A\|^{-t})^{(kq - l + q/2)/(q-1)}$, if $k > l/q$.

Theorem [1]. If $k > l/q$, then the equation (2) has a solution $\alpha = \alpha(\delta)$ for every $b > 0$ and $\|x_{\alpha(\delta)} - x_*\| \to 0$ as $\delta \to 0$, provided that $\|y - y_*\|/\delta \leq \text{const}$ in the process $\delta \to 0$. If $k = l/q$, $\quad b \geq b_0(q, l, k)$ and $\|y - y_*\| \leq \delta$ then the equation (2) has the unique solution $\alpha = \alpha(\delta)$ and $\|x_{\alpha(\delta)} - x_*\| \to 0$ as $\delta \to 0$.

References

1. U. Hämarik, R. Palm, and T. Raus, A family of rules for parameter choice in Tikhonov regularization of ill-posed problems with inexact noise level. *Journal of Computational and Applied Mathematics*, **236**, 2146 - 2157 (2012).