Stability Estimates and Convergent Numerical Method for Thermoacoustic Tomography with an Arbitrary Elliptic Operator

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Recently the author has obtained, *for the first time*, logarithmic stability estimates and a convergent numerical method for thermoacoustic tomography with an arbitrary elliptic operator [1], i.e. for the equation

$$u_{tt} = L(x)\,u := \sum_{i,j=1}^{n} a_{i,j}(x)\,\partial^2_{x_i x_j} u + \sum_{j=1}^{n} b_j(x)\,\partial_{x_j} u + a(x)\,u, (x,t) \in \mathbb{R}^n \times (0,T), \tag{1}$$

$$u(x,0) = f(x), u_t(x,0) = 0, \tag{2}$$

where $L(x)$ is uniformly elliptic operator in $\mathbb{R}^n$. This was done for both complete and incomplete data collection cases. Results of [1] will be presented in the talk.

Thermoacoustic tomography arises in medical imaging. From the mathematical standpoint this problem can be formulated as follows. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $S_T = \partial\Omega \times (0,T)$. Suppose that $f(x) = 0$ and $L(x) = \Delta$ outside of Ω.Determine the function $f(x)$ for $x \in \Omega$ assuming that the function $\varphi(x,t)$ is known,

$$u\mid_{S_T} = \varphi(x,t). \tag{3}$$

Solving equation (1) for $(x,t) \in (\mathbb{R}^n \diagdown \Omega) \times (0,T)$ with initial conditions $u(x,0) = u_t(x,0) = 0, x \in \mathbb{R}^n \diagdown \Omega$ and boundary condition (3), one can uniquely and stably determine the Neumann boundary condition $\psi(x,t)$,

$$\partial_n u\mid_{S_T} = \psi(x,t). \tag{4}$$

In the incomplete data collection case in [1] $\Omega = \{x_1 > 0\}, S_T = \{x_1 = 0\} \times (0,T)$. Considering even extension $u(x,-t) := u(x,t), t \in (0,T)$, one obtains equation (1) in

$Q_T^{\pm} = \Omega \times (-T, T)$ with both Dirichlet and Neumann boundary conditions (3), (4) given at $S_T^{\pm} = \partial\Omega \times (-T, T)$.

In the works [2-8] in 1991-2008 the author and his coauthors have Lipschitz stability estimates for the case when

$$L(x,t)\,u = c^2(x)\,\Delta u + \sum_{j=1}^{n} b_j(x,t)\,\partial_{x_j} u + a(x,t)\,u. \tag{5}$$

In particular, hyperbolic inequalities with the lateral Cauchy data at $S_T^{\pm}$ were considered in [3,5]

$$\left|u_{tt} - c^2(x)\,\Delta u\right| \leq A\left(\left|\nabla_{x,t} u\right| + |u| + |g(x,t)|\right), A = const > 0, (x,t) \in Q_T^{\pm}. \tag{6}$$

These estimates were

$$\|u\|_{H^1\left(Q_T^{\pm}\right)} \leq C\left(\|\varphi\|_{H^1\left(S_T^{\pm}\right)} + \|\psi\|_{L_2\left(S_T^{\pm}\right)} + \|g\|_{L_2\left(Q_T^{\pm}\right)}\right). \tag{7}$$

Hence, the trace theorem implies with a different constant C

$$\|u(x,0)\|_{L_2(\Omega)} := \|f\|_{L_2(\Omega)} \leq C\left(\|\varphi\|_{H^1\left(S_T^{\pm}\right)} + \|\psi\|_{L_2\left(S_T^{\pm}\right)} + \|g\|_{L_2\left(Q_T^{\pm}\right)}\right).$$

In addition, convergence of the quasi-reversibility method was proven [2,5,7]. Numerical results have consistently demonstrated a high degree of stability of this method [3,7] with up to 50% noise in the data [8]. The key tool of works [2-8] is an idea connected with Carleman estimates. Although in some works [2,6] the assumption was that $c \equiv 1$ in (5), (6), it is clear from the first publications [2,3] that as soon as the principal part of the hyperbolic operator is such that the Carleman estimate holds, the method of [2] works. This thought is reflected in the proof of Theorem 3.4.8 of the book [9].

However, results of [2-8] are proven under a restrictive condition on the coefficient $c(x)$,

$$\left(x - x_0, \nabla\left(c^{-2}(x)\right)\right) + \alpha \geq 0, x \in \overline{\Omega} \text{ for an } \alpha = const. > 0 \tag{8}$$

for a point x_0. Clearly $c(x) \equiv 1$ satisfies (8). In particular, Lipschitz stability (7) for inequality (6) with condition (8) was obtained in [5] and for the equation (5) with condition (8) in [7].

Condition (8) was imposed because in the hyperbolic case the Carleman estimate is known only for the operator $\partial_t^2 - c^2(x)\,\Delta$ with $c(x)$ satisfying (8). Other known results [10,11], which were published later than [2], impose the non-trapping condition on $c(x)$ and do not work for hyperbolic PDEs with lower order terms as in (5). Since the non-trapping condition cannot be directly analytically verified, then it is equally restrictive with condition (8). In addition, a small variation of (8) guarantees non-trapping, see formula (3.24) in [12].

Prior to [1] both stability estimates and convergent numerical methods were not published neither for the case of an arbitrary elliptic operator in (1) nor for the case when no restrictive conditions are imposed on $c(x)$ in (5). Also, convergent numerical methods were not developed for the case of incomplete data collection even with $c(x) \equiv 1$. Although in [1] $T = \infty$, it was shown there that this is not a serious restriction for many applications (see second and third Remarks 3.1 in [1]).

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