Local volatility as an inverse problem

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Since Dupire published his celebrated paper [1] in 1994, his local volatility model has become one of the most extensively used models in derivatives pricing across all asset classes. In the case of an equity stock or index S, the price dynamics in the local volatility model under the risk neutral measure are given as

$$\mathrm{d}S_t = (r_t - q_t)S_t\,\mathrm{d}t + \sigma(t, S_t)S_t\,\mathrm{d}W_t \tag{1}$$

where W_t is a Brownian motion, r_t is the risk free interest rate and q_t is a continuous dividend yield at time t. The squared local volatility σ^2 gives the instantaneous variance of the logarithm of S as a deterministic function of the time t and the spot value S_t. Dupire showed that under the dynamics (1), the prices of call options $C(T,K)$ of time to maturity T, strike K and a given value of the spot S_0 at time $t = 0$, can be related through the following parabolic partial differential equation,

$$\begin{aligned}
\partial_T C(T,K) &= \frac{1}{2}\sigma^2(T,K)K^2\partial_{KK}C(T,K) - q_T C(T,K) - (r_T - q_T)K\partial_K C(T,K)\\
&\quad (T,K) \in \mathbb{R}_+ \times \mathbb{R}_+,\\
C(0,K) &= (S_0 - K)_+,\ C(T,0) = S_0\,\mathrm{e}^{-\int_0^T q_t \mathrm{d}t},\ C(T,\infty) = 0
\end{aligned} \tag{2}$$

from which the volatility function can be expressed in terms of option prices as

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) + q_T C(T,K) + (r_T - q_T)K\partial_K C(T,K)}{\frac{1}{2}K^2\partial_{KK}C(T,K)}. \tag{3}$$

The model's popularity stems from this simple relation between option prices and the volatility function: given a surface of option prices $C : (T,K) \to \mathbb{R}_+$ that is once differentiable in T and twice differentiable in K, the function σ can be retrieved from a mere differentiation of C. An appealing consequence of that observation is of course that with σ chosen according to (3), the partial differential equation (2) tells us that an asset with the dynamics (1) will match all option prices $C(T,K)$.

The simplicity of the expression (3), however, is somewhat illusive. In practice, market prices on options are not given as continuous, smooth surfaces, but as discrete values that can not obviously be seen to be sampled from a differentiable function. So even though the equation (3) gives a seemingly easy way of constructing the function σ from option prices, this is difficult in real life where we can only observe option prices at a finite number of maturity-strike pairs.

In this talk we will illustrate that this inverse problem of choosing a local volatility function that makes the model replicate observable market prices can be succesfully handled as an

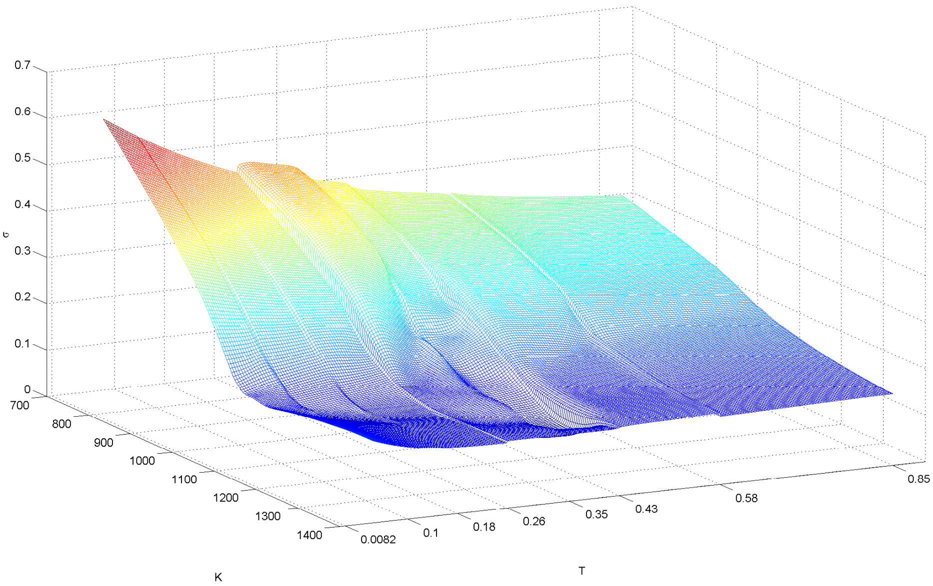

Figure 1: Local volatility for OMXS30 at 17:05 pm, August 14, 2012.

optimal control problem. A general optimal control problem for a function constrained to follow Dupire's equation on an interval $[0, \bar{T}]$ can be stated as

$$\begin{aligned} &\min_{\sigma \in \Sigma} \int_0^{\bar{T}} h(T, C(T))\, \mathrm{d}T \\ \text{subject to:}\quad &C,\, \sigma \text{ satisfy (2)}, \end{aligned} \tag{4}$$

where Σ is some space of functions on $[0, \bar{T}] \times \mathbb{R}_+$. If h is chosen as a distance between C and some observed market prices $\bar{C}$, then a volatility function σ that satisfies (4) can be seen as an optimal control that "steers" the model option prices C as close as possible to observed market data under the constraint that C satisfies Dupire's equation (2).

The algorithms we develop are based on the techniques from [2] for solving optimal control problems of the type (4) through a regularized version of the corresponding Hamiltonian system. In order to get satisfactory results when working with actual market data, we will also use techniques based on the use of so called affine stochastic volatility models [3] in order to calculate a good initial guess for the local volatility σ.

In Figure 1 we give an example of a local volatility model obtained with our method for the OMXS30 stock index as of August 14, 2012. In our presentation, we will also show how well the prices C resulting from a solution to (2) with our local volatility function σ replicate the market prices we use as data.

References

[1] Bruno Dupire. Pricing with a smile. *Risk*, pages 18–20, January 1994.

[2] Mattias Sandberg and Anders Szepessy. Convergence rates of symplectic Pontryagin approximations in optimal control theory. *ESAIM: Mathematical Modelling and Numerical Analysis*, 40:149–173, January 2006.

[3] Darrel Duffie, Jun Pan, and Kenneth Singleton. Transform Analysis and Asset Pricing for Affine Jump-Diffusions. *Econometrica*, 68(6):1343–1376, November 2000.