Reconstructing a Function from its V-line Radon Transform in a Disc

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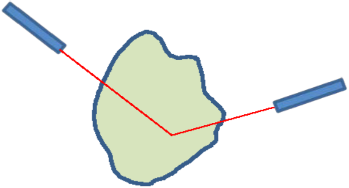
Single Scattering Optical Tomography (SSOT)



- Uses light, transmitted and scattered through an object, to determine the interior features of that object.
- If the object has moderate optical thickness it is reasonable to assume the majority of photons scatter once.
- Using collimated emitters/receivers one can measure the intensity of light scattered along various broken rays.
- Need to recover the spatially varying coefficients of light absorption and/or light scattering.

Florescu, Schotland and Markel (2009, 2010, 2011)

Mesoscopic Radiative Transport.

$$[\hat{\mathbf{s}} \cdot \nabla + \mu_a(\mathbf{r}) + \mu_s(\mathbf{r})]\, I(\mathbf{r}, \hat{\mathbf{s}}) = \mu_s(\mathbf{r}) \int A(\hat{\mathbf{s}}, \hat{\mathbf{s}}')\, I(r, \hat{\mathbf{s}}')\, d\hat{\mathbf{s}}'. \quad (1)$$

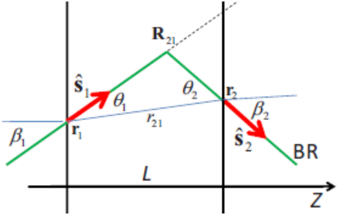
$I(\mathbf{r}, \hat{\mathbf{s}})$ is the light intensity at point $\mathbf{r}$ in the direction $\hat{\mathbf{s}}$.

$\mu_a(\mathbf{r})$ and $\mu_s(\mathbf{r})$ are the absorption and scattering coefficients.

$A(\hat{\mathbf{s}}, \hat{\mathbf{s}}')$ is the probability that a photon travelling in the direction of $\hat{\mathbf{s}}$ is scattered in the direction of $\hat{\mathbf{s}}'$.

$$I(\mathbf{r}, \hat{\mathbf{s}}) = I_0(\mathbf{r}, \hat{\mathbf{s}}), \;\; \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}(\mathbf{r}) < 0, \;\; \mathbf{r} \in \partial V. \quad (2)$$

Florescu, Schotland and Markel (2009, 2010, 2011)



Using Born approximation (1), (2) can be reduced to an integral geometry problem:

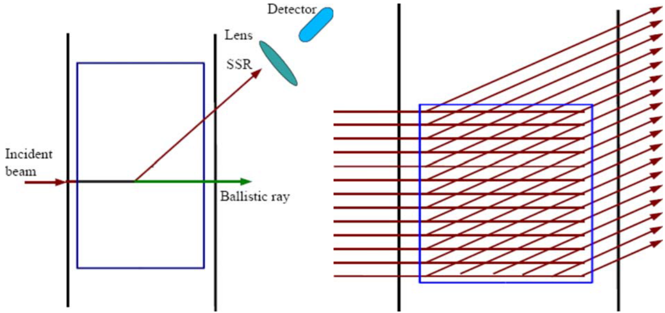
$$\phi(\mathbf{r}_2, \hat{\mathbf{s}}_2, \mathbf{r}_1, \hat{\mathbf{s}}_1) = \int\limits_{\mathrm{BR}(\mathbf{r}_2, \hat{\mathbf{s}}_2, \mathbf{r}_1, \hat{\mathbf{s}}_1)} \mu_t[\mathbf{r}(l)]dl - \ln\left[\frac{\mu_s(\mathbf{R}_{21})}{\bar{\mu}_s}\right], \qquad (3)$$

where

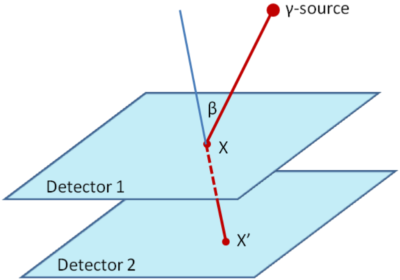
$$\phi(\mathbf{r}_2, \hat{\mathbf{s}}_2, \mathbf{r}_1, \hat{\mathbf{s}}_1) = -\ln\left[\frac{r_{21}\sin\theta_1\sin\theta_2 \int I_s(\mathbf{r}_2, \hat{\mathbf{s}}_2, \mathbf{r}_1, \hat{\mathbf{s}}_1)d\varphi_{\hat{\mathbf{s}}_2}}{I_0\,\bar{\mu}_s\,A(\hat{\mathbf{s}}_2, \hat{\mathbf{s}}_1)}\right]. \qquad (4)$$

Florescu, Schotland and Markel (2009, 2010, 2011)

So if the scattering coefficient is known, then the reconstruction of the absorption coefficient is reduced to inversion of a generalized Radon transform integrating along the broken rays.



Compton Scattering



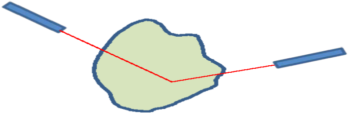
$$\cos\beta = 1 - \frac{mc^2\,\Delta E}{(E-\Delta E)E}$$

- R. Basko, G. Zeng, and G. Gullberg (1997, 1998)
- M. Nguyen, T. Truong, et al (2000's)

Broken Rays and Broken Geodesics

- M. Hubenthal
- J. Ilmavirta
- M. Lassas
- M. Salo
- G. Uhlmann
- . . .

V-line Radon Transform (VRT) in 2D



Definition

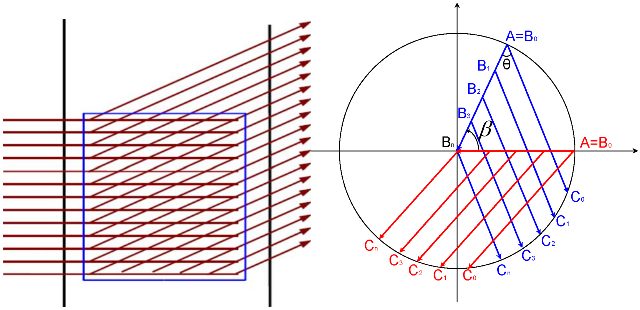
The V-line Radon transform of function $f(x, y)$ is the integral

$$\mathcal{R}f(\beta, t) = \int\limits_{BR(\beta,t)} f\, ds, \qquad (5)$$

of f along the broken ray $BR(\beta, t)$ with respect to line measure ds.

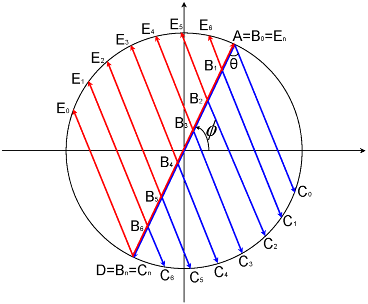
The problem of inversion is over-determined, so it is natural to consider a restriction of $\mathcal{R}f$ to a two-dimensional set.

Geometry: Slab vs Disc



- Available directions
- Stability of reconstruction
- Hardware implementation (?)

Full Data (G.A. 2012)



Theorem

If $f(x, y)$ is a smooth function supported in the disc $D(0, R\sin\theta)$, then f is uniquely determined by $\mathcal{R}f(\phi, d)$, $\phi \in [0, 2\pi]$, $d \in [0, 2R]$.

Inversion Formula

$$\widetilde{\mathcal{R}}f(\psi_\phi, t_d) = \mathcal{R}f(\phi, d) + \mathcal{R}f(\phi + \pi, 2R - d) - \mathcal{R}f(\phi, 2R), \quad (6)$$

for all values $\phi \in [0, 2\pi]$ and $d \in [0, 2R]$.

$$f(x,y) = \frac{1}{4\pi} \int_0^{2\pi} \mathcal{H}\left(\widetilde{\mathcal{R}}f\,'_t\right)(\psi, x\cos\psi + y\sin\psi)\; d\psi \qquad (7)$$

where $\mathcal{H}$ is the Hilbert transform defined by

$$\mathcal{H}h(t) = -\frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \operatorname{sgn}(r)\, \widehat{h}(r)\, e^{irt}\, dr. \qquad (8)$$

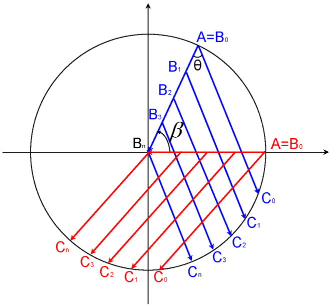
and $\widehat{h}(r)$ is the Fourier transform of $h(t)$, i.e.

$$\widehat{h}(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(t)\, e^{-irt}\, dt. \qquad (9)$$

Inversion Formula

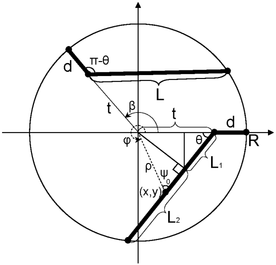
- Issues with the support
- Interior problem
- Other methods without loss of information
- Rotation invariance

Partial Data (G.A., S. Moon 2013)



Theorem

If $f(x, y)$ is a smooth function supported in the disc $D(0, R)$, then f is uniquely determined by $\mathcal{R}f(\phi, d)$, $\phi \in [0, 2\pi]$, $d \in [0, R]$.



Denote $g(\beta, t) := \mathcal{R}f(\beta, t)$.

$$f(\phi, \rho) = \sum_{n=-\infty}^{\infty} f_n(\rho)\, e^{in\phi}, \qquad g(\beta, t) = \sum_{n=-\infty}^{\infty} g_n(t)\, e^{in\beta},$$

where the Fourier coefficients are given by

$$f_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi, \rho)\, e^{-in\phi} d\phi, \qquad g_n(t) = \frac{1}{2\pi}\int_0^{2\pi} g(\beta, t)\, e^{-in\beta} d\beta.$$

Inversion Formula

$$\mathcal{M}f_n(s) = \frac{\mathcal{M}g_n(s-1)}{1/(s-1) + \mathcal{M}h_n(s-1)}, \quad \Re(s) > 1 \tag{10}$$

where $\mathcal{M}F$ denotes the Mellin transform of function F

$$\mathcal{M}F(s) = \int_0^\infty p^{s-1} F(p)\, dp,$$

and h_n is some fixed function. Hence for any $t > 1$ we have

$$f_n(\rho) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{t-Ti}^{t+Ti} \rho^{-s} \frac{\mathcal{M}g_n(s-1)}{1/(s-1) + \mathcal{M}h_n(s-1)}\, ds. \tag{11}$$

If $1 < t < \frac{1}{\sin\theta}$ then

$$h_n(t) = (-1)^n e^{in\psi(t)} \frac{1 + t\cos[\psi(t)] + t^2 \sin[\psi(t)] \frac{\sin\theta}{\sqrt{1-t^2\sin^2\theta}}}{\sqrt{1+t^2+2t\cos(\psi(t))}}$$

$$- e^{in[2\theta-\psi(t)]} \frac{1 - t\cos[2\theta-\psi(t)] + t^2 \sin[2\theta-\psi(t)] \frac{\sin\theta}{\sqrt{1-t^2\sin^2\theta}}}{\sqrt{1+t^2-2t\cos[2\theta-\psi(t)]}},$$

$$h_n(t) = (-1)^n e^{in\psi(t)} \frac{1 + t\cos[\psi(t)] + t^2 \sin[\psi(t)] \frac{\sin\theta}{\sqrt{1-t^2\sin^2\theta}}}{\sqrt{1+t^2+2t\cos[\psi(t)]}}, \quad 0 < t \leq 1$$

and $h_n(t) \equiv 0$, for all $t > \frac{1}{\sin\theta}$. Here $\psi(t) = \arcsin(t\sin\theta) + \theta$.

Current Work and Open Problems

- Numerical implementation
- Stability and microlocal analysis
- Range description
- Over-determined setups

Thanks for Your Attention!

- Ambartsoumian, G., Inversion of the V-line Radon transform in a disc and its applications in imaging, *Computers and Mathematics with Application*, (2013).
- Ambartsoumian, G., Moon, S., A series formula for inversion of the V-line Radon transform in a disc, *Computers and Mathematics with Application*, (2012).
- Florescu, L., Schotland, J.C., Markel, V.A., Single scattering optical tomography, *Phys. Rev. E* **79**, 036607 (2009).
- Florescu, L., Markel, V.A., Schotland, J.C., Single-scattering optical tomography: Simultaneous reconstruction of scattering and absorption, *Phys. Rev. E* **81**, 016602 (2010).
- Florescu, L., Markel, V.A., Schotland, J.C., Inversion formulas for the broken-ray Radon transform, *Inverse Problems* **27**, 025002 (2011).