An inverse problem for the wave equation with one measurement

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Joint work with

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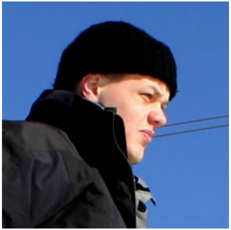
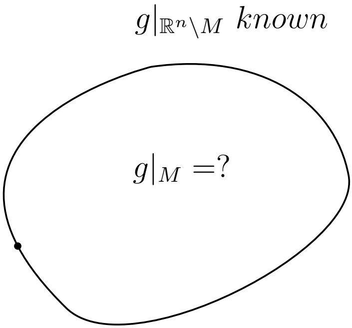


Lauri Oksanen

An inverse problem for the wave equation with one measurement and the pseudorandom noise.

in Analysis and PDE 5 (2012), 887-912.

An inverse problem with a single measurement



$$(\partial_t^2 - \Delta_g)u = f \quad \text{in } \mathbb{R}^n \times (0,T),$$
$$u|_{t<0} = 0.$$

Assume $g|_{\mathbb{R}^n\setminus M}$ is known.

Measure $u|_{\partial M\times(0,T)}$ for a single source f supported on ∂M.

How to choose f to get useful information about $g|_M$?

Single vs. many measurements

The problem of **many measurements**:

find g given the hyperbolic Dirichlet-to-Neumann map (DN-map).

$\Longrightarrow$ g is determined up to an isometry [Belishev, Kurylev '92 & Tataru '95].

The problem is overdetermined, since, formally,

$$\dim(\ker(\Lambda_{DN})) = 2n - 1 \qquad \text{and} \qquad \dim(g) = n.$$

Single measurement: $u|_{\partial M \times (0,T)}$ depends on n variables. The problem here is formally determined.

Pseudo-random source and the measurement

We let

- $(x_j)_{j=1}^{\infty} \subset \partial M$ be a dense sequence of distinct points in ∂M and
- $(a_j)_{j=1}^{\infty} \subset \mathbb{R}$ such that $\sum_{j=1}^{\infty} |a_j| < \infty$.

We define the pseudo-random source by

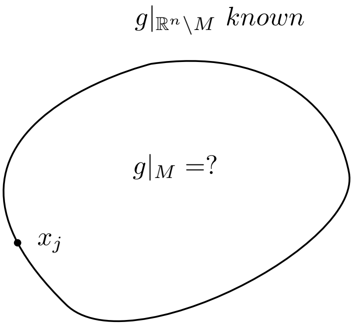
$$f(x,t) = \sum_{j=1}^{\infty} a_j \delta(x - x_j, t) = \sum_{j=1}^{\infty} a_j \delta_j, \quad (x.t) \in \mathbb{R}^{n+1}.$$

$\Longrightarrow$ For any $p \in (1, \frac{n}{n-1})$ and $\epsilon > 0$, f satisfies

$$f \in H^{-1}(-\epsilon, \epsilon) \otimes H_p^{-1}(\mathbb{R}^n).$$

- Compressed in time.
- In practise, could be imitated by a random point process.

Some notations and assumptions



$$(\partial_t^2 - \Delta_g)u = \sum_{j=1}^{\infty} a_j \delta_j \quad \text{in } \mathbb{R}^n \times (T_0, T),$$

$$u|_{t<T_0} = 0, T_0 < 0.$$

Here $g(x) = (g_{jk}(x))_{j=1}^n$ is a smooth Riemannian metric:

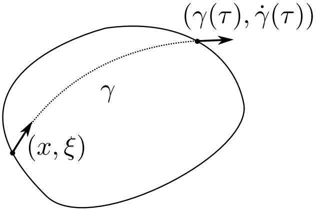
- $|g| = \det(g)$,
- $g^{-1} = (g^{jk}(x))_{j=1}^n$ and
- $\Delta_g u = |g|^{-1/2} \sum_{j,k=1}^n \partial_j \left(g^{jk} |g|^{1/2} \partial_k u \right).$

(i) M is open and bounded and the boundary ∂M is smooth.

(ii) g is smooth and there are $c_1, c_2 > 0$ s.t.

$$c_1 |\xi|^2 \leq \sum_{j.k=1}^{n} g_{jk}(x) \xi^j \xi^k \leq c_2 |\xi|^2, \quad x, \xi \in \mathbb{R}^n.$$

The scattering relation and the main result



Denote by $\gamma_{x,\xi}$ the geodesic γ satisfying

$$\gamma(0) = x, \quad \dot\gamma(0) = \xi,$$

and define the exit time

$$\tau(x, \xi) := \inf\{t \in (0, \infty];\ \gamma_{x,\xi} \in \partial M\}.$$

We define the scattering relation Σ on the set of non-trapped inward pointing unit vectors $D(\Sigma) = \{(x, \xi) \in \partial_- SM;\ \tau(x, \xi) < \infty\}$ by

$$\Sigma(x, \xi) := (\gamma(\tau), \dot\gamma(\tau), \tau), \quad \gamma = \gamma_{x,\xi},\ \tau = \tau(x, \xi).$$

Theorem

Let $a_j = 2^{-2^j}$. If $T > \sup_{\partial_- SM} \tau$, then $u|_{\partial M \times (0,T)}$ determines Σ. If there are trapped geodesics, we must take $T = \infty$ to determine $D(\Sigma)$ and Σ.

On the scattering relation

The scattering relation Σ is known to determine g (up to an isometry) in the following classes:

- non-trapping real analytic metrics [Vargo '10],
- non-trapping metrics close to an analytic metric [Stefanov, Uhlmann '09].

If $(\overline{M}, g)$ is simple, Σ determines g using boundary rigidity results known in the following cases:

- dimension $n = 2$ [Pestov, Uhlmann '05],
- metrics close to the Euclidean metric [Burago, Ivanov '10].

On the proof: continuation into the exterior domain

Solve u in the exterior domain:

$$(\partial_t^2 - \Delta_g)u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{M} \times (T_0, T),$$
$$u|_{\partial M \times (T_0,T)} = \text{measurement},$$
$$u|_{t=T_0} = \partial_t u|_{t=T_0} = 0.$$

Take w to be any smooth solution of

$$(\partial_t^2 - \Delta_g)w = 0 \quad \text{in } \mathbb{R}^n \times (T_0, t_0),$$

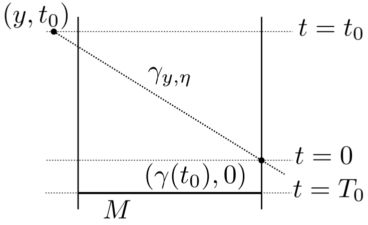
with $\text{supp}(w|_{t=t_0}), \text{supp}(\partial_t w|_{t=t_0}) \subset \mathbb{R}^n \setminus \overline{M}$.

Then we can determine the right hand side of

$$\int_{T_0}^{t_0} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} a_j \delta_j \right) w \; dt dV = \int_{\mathbb{R}^n} \partial_t u \; w - u \; \partial_t w \; dV|_{t=t_0},$$

for any $t_0 \leq T$.

Gaussian beams



The Gaussian beam solution $w = w_{\epsilon,y,\eta}$ of

$$
\begin{aligned}
&(\partial_t^2 - \Delta_g)w = 0 \quad \text{in } \mathbb{R}^n \times (T_0,t_0),\\
&w(x,t_0) = \chi_y(x)W(0,x),\\
&\partial_t w(x,t_0) = -\chi_y(x)\partial_t W(0,x)
\end{aligned}
$$

satisfies $w(x,t_0-t) = O(\epsilon)$, $x \neq \gamma_{y,\eta}(t)$.

By construction W solves the wave equation up to an error of order ϵ^N for a given N and is of form

$$e^{i\theta(x,t)/\epsilon}a_\epsilon(x,t).$$

The phase function θ satisfies

$$\theta(\gamma(t),t) = 0, \quad \operatorname{Im}\theta(x,t) \geq c_W(t)d_g(x,\gamma(t))^2, \quad c_W > 0,$$

where $\gamma = \gamma_{y,\eta}$ is the geodesic with initial data (y,η).

To construct W(0,x) we need to know the metric g only in a neighborhood of x.

Indicator function

We write

$$I(y,\eta,t_0) := \lim_{\epsilon\to 0} \int_{T_0}^{t_0} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} a_j \delta_j \right) w_{\epsilon,y,\eta} \, dt dV.$$

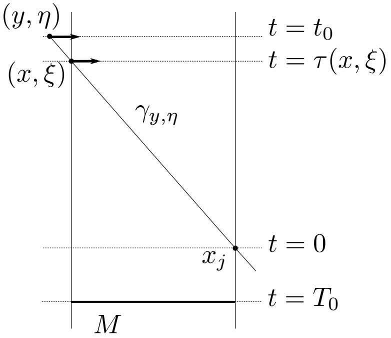
Lemma

One can show that

$$I(y,\eta,t_0) = \begin{cases} a_j b(y,\eta), & \gamma_{y,\eta}(t_0) = x_j, \\ 0, & \gamma_{y,\eta}(t_0) \neq x_j, \ \forall j. \end{cases}$$

Here b is a non-vanishing smooth function depending on $g|_M$.

Exit times



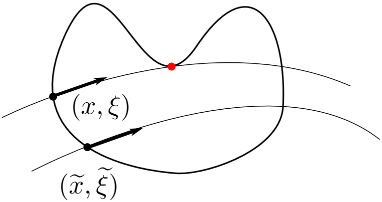
Denote $\Omega := \mathbb{R}^n \setminus M$ and let $(y,\eta) \in S\Omega$. We know, if $\gamma_{y,\eta}(t_0) = x_j$ for some j.

As $g|_\Omega$ is known, we get the entering point and direction (x,ξ) and also the entering time of $\gamma_{y,\eta}$.

By varying t_0 and (y,η), we get $\tau(x,\xi)$ for all geodesics which exit through some source point x_j.

We want to use density $(x_j)_{j=1}^\infty \subset \partial M$, to get $\tau(x,\xi)$ for all geodesics passing through M.

Exit times (cont.)



- If $\gamma_{x,\xi}$ intersects ∂M tangentially, τ can be discontinuous at (x,ξ).
- However, if $\gamma_{x,\xi}(t)$ is in the interior of M, then $\gamma_{\widetilde{x},\widetilde{\xi}}(t)$ is also in the interior for nearby $(\widetilde{x},\widetilde{\xi})$.

- As a limit, intersections may appear, but not disappear.
- This means that τ is lower semi-continuous on $\partial_- SM$.
- We get $\tau(x,\xi)$ for all $(x,\xi) \in \partial_- SM$ using density of the source points.

Exit point

So far we have only used the information

$$I(y,\eta,t_0)\begin{cases}\neq 0, & \gamma_{y,\eta}(t_0)=x_j,\\ =0, & \gamma_{y,\eta}(t_0)\neq x_j,\ \forall j.\end{cases}$$

Let us show next that the factor $b(y,\eta)$ can be separated in

$$I(y,\eta,t_0)=a_j b(y,\eta), \quad \gamma_{y,\eta}(t_0)=x_j,$$

assuming that the weights of the point sources are $a_j = 2^{-2^j}$.

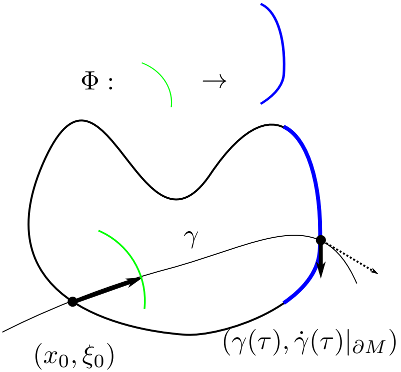
Exit point (cont.)

- Let (y_0, η_0) and t_0 be such that $\gamma_{y_0,\eta_0}(t_0) = x_{j_0}$ for some unknown j_0. We want to find j_0.
- Choose (y_k, η_k) and t_k converging to (y_0, η_0) and t_0 such that
  - $\gamma_{y_k,\eta_k}(t_k) = x_{j_k}$ for some unknown j_k
  - the known numbers $2^{-2^{j_k}} b(y_k, \eta_k)$ converge to zero.
- Then $j_k \to \infty$, and

$$\log_2 \left(2^{-2^{j_k}} |b(y_k, \eta_k)| \right) = -2^{j_k} + \log_2 |b(y_k, \eta_k)|.$$

- $\log_2 |b(y_k, \eta_k)|$ becomes asymptotically a small perturbation in the grid $(-2^j)_{j=1}^{\infty}$. Thus the limit perturbation $\log_2 |b(y_0, \eta_0)|$ is determined.
- As $2^{-2^{j_0}} b(y_0, \eta_0)$ is known we can solve for j_0, whence the exit point x_{j_0} is determined.

Exit direction



- If $\tau = \tau(x_0, \xi_0) < \infty$, $\gamma = \gamma_{x_0,\xi_0}$ is transverse to ∂M and $\gamma(\tau)$ is not conjugate to x_0 along γ, then the function

$$\Phi : \xi \mapsto \gamma_{x_0,\xi}(\tau(x_0, \xi))$$

is a local diffeomorphism $S_{x_0}M \rightarrow \partial M$ near ξ_0, and

$$grad_{\partial M}(\tau(x_0, \Phi^{-1}(z))|_{z=\gamma(\tau)} = \dot{\gamma}(\tau)^\top|_{\partial M}.$$

- Transversality is a generic property.
- As conjugate points are discrete on γ, we may choose $(y, \eta) \in S\Omega$ lying on γ, not conjugate to $\gamma(\tau)$, and employ the same construction.

Conclusions

- We have shown that $u|_{\partial M\times(0,T)}$ determines
  - the exit time τ,
  - the exit point $\gamma(\tau)$,
  - the exit direction $\dot{\gamma}(\tau)$.
- Thus $u|_{\partial M\times(0,T)}$ determines the scattering relation Σ.
- Σ determines the metric g in some classes of metrics.
- Thus the formally determined inverse problem of finding n-dimensional unknown g given n-dimensional data $u|_{\partial M\times(0,T)}$ is solvable in some classes of metrics.