Stability Estimates and Convergent Numerical Method for Thermoacoustic Tomography with an Arbitrary Elliptic Operator

Michael V. Klibanov

Department of Mathematics and Statistics,
University of North Carolina at Charlotte,
Charlotte, NC 28223, USA. Email: mklibanv@uncc.edu

All theorems below are only brief outlines of results: for brevity of this presentation. Detailed formulations can be found in the paper:

Inverse Problems, 29, 25014, 2013.

Inverse problem of thermoacoustic tomography

• In thermoacoustic tomography a short radio frequency pulse is sent in a biological tissue. Some energy is absorbed. Malignant legions absorb more energy than healthy ones. Then the tissue expands and radiates a pressure wave.

Inverse Problem. Let $\Omega \subset \mathbb{R}^3$, $\partial\Omega \in C^4$ be a bounded domain, $Q_T = \partial\Omega \times (0, T), S_T = \partial\Omega \times (0, T)$.

$$\begin{aligned} u_{tt} &= c^2(x)\,\Delta u, x \in \mathbb{R}^3, t \in (0, T), &(1)\\ u(x,0) &= f(x), u_t(x,0) = 0. &(2)\\ f(x) &= 0, c(x) = 1, x \in \mathbb{R}^n \diagdown \Omega. &(3) \end{aligned}$$

Given the function $g(x,t)$,

$$u \mid_{S_T} = g(x,t), \qquad (4)$$

find the initial condition $f(x)$.

Obtaining Neumann boundary condition

Step 1 (elementary). Find the normal derivative at $h(x,t) = \partial_\nu u \,|_{S_T}$. Solve the initial boundary value problem

$$\begin{aligned} u_{tt} &= \Delta u, x \in \mathbb{R}^3 \diagdown \Omega, t \in (0,T), \\ u(x,0) &= 0, u_t(x,0) = 0, x \in \mathbb{R}^n \diagdown \Omega, \\ u &\mid \, S_T = g(x,t). \end{aligned}$$

Hence,

$$\|h\|_{L_2(S_T)} \leq C \|g\|_{H^2(S_T)}. \tag{5}$$

Published Results

1. Lipschitz stability via Carleman estimates for hyperbolic equations and inequalities.

Klibanov and Malinsky, 1991 **(the first result)**; Kazemi and Klibanov, 1993; Klibanov and Timonov (book), 2004; Klibanov, 2005; Lasiecka, Triggiani and Zhang, 1999, 2004 (two papers; applications in the control theory); Isakov (book, 2006); Romanov 2006 (two papers); Clason and Klibanov, 2007;Klibanov, survey: "Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems", Journal of Inverse and Ill-Posed Problems, published online, 2013; preprint is available on arxiv.

Let $x_0 \in \Omega$,

$$\left(x - x_0, \nabla \left(c^{-2}\left(x\right)\right)\right) \geq \alpha = const. > 0, \forall x \in \overline{\Omega}. \qquad (6)$$

Particular case: $c\left(x\right) \equiv 1$. A slight modification of (6) implies non-trapping.

Hyperbolic inequality

$$\begin{aligned} \left| w_{tt} - c^2(x)\,\Delta w \right| &\leq A\left[|\nabla w| + |w_t| + |w| + |p| \right] \text{ in } Q_T, \quad (7) \\ w_t(x,0) &= 0. \end{aligned}$$

Then

$$\|w\|_{H^1(Q_T)} \leq C\left[\left\| w \mid_{S_T} \right\|_{H^1(S_T)} + \left\| \partial_\nu w \mid_{S_T} \right\|_{L_2(S_T)} + \|p\|_{L_2(Q_T)} \right]. \quad (8)$$

The trace theorem (5) and (8) imply that for thermoacoustic tomography

$$\|f\|_{L^2(Q_T)} \leq C\,\|g\|_{H^2(S_T)}.$$

$T = T(c)$ is sufficiently large. In the case $c(x) \equiv 1$,
$T > diam(\Omega)/2$.

2. Numerical Methods.

Quasi-Reversibility of Lattes and Lions (1969), convergence via Lipschitz stability: Klibanov and Malinsky, 1991 (theory). Numerics and convergence: Klibanov and Rakesh, 1992; Clason and Klibanov, 2007; Klibanov, Kuzhuget, Kabanikhin and Nechaev, 2008. Agranovsky and Kuchment, 2007.

3. Explicit reconstruction formulae for the case of the wave operator.

Good performance of numerical methods: Finch, Patch and Rakesh, 2004; Finch, Haltmeier and Rakesh, 2007; Kunyansky, 2008; Kunyansky and Kuchment, 2008 (survey). Good numerical performances.

• However, in all past publications some restrictive conditions were imposed on the function $c(x)$, e.g. (6).

• The case of a general elliptic operator $L(x)$ in $u_{tt} = L(x)u$ was not considered.

• Numerical methods for the case of a general elliptic operator $L(x)$ were not developed.

$$Lu = \sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i x_j} + \sum_{j=1}^{n} b_j(x) u_{x_j} + c(x) u, x \in \mathbb{R}^n, \quad (9)$$

$$\mu_1 |\eta|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x) \eta_i \eta_j \leq \mu_2 |\eta|^2, \forall x \in \mathbb{R}^n, \forall \eta \in \mathbb{R}^n; \quad (10)$$

$$\mu_1, \mu_2 = const. > 0, \quad (11)$$

$$f \in H^{s+5}(\mathbb{R}^n), a_{i,j}, b_j, c \in C^{s+3}(\mathbb{R}^n), s = \left[\frac{n+1}{2}\right] \quad (12)$$

Cauchy problem

$$u_{tt} = Lu, x \in \mathbb{R}^n, t \in (0, \infty), \quad (13)$$

$$u(x,0) = f(x), u_t(x,0) = 0. \quad (14)$$

Inverse Problem 1 (IP1, Complete Data Collection). *Assume that the function $f(x)$ is unknown. Determine this function, assuming that the following function $\varphi_1(x,t)$ is known*

$$u\mid_{S_\infty}=\varphi_1(x,t). \tag{15}$$

Let

$$\Omega\subset\{x_1>0\}, P=\{x_1=0\}, P_\infty=P\times(0,\infty).$$

Inverse Problem 2 (IP2, Incomplete Data Collection).
Assume that the function $f(x)$ *is unknown. Determine this function, assuming that the following function* $\varphi_2(x,t)$ *is known*

$$u\mid_{x\in P_\infty}=\varphi_2(x,t). \tag{16}$$

Reznickaya transform (1974)

$$\mathcal{L}u=v(x,t)=\frac{1}{\sqrt{\pi t}}\int\limits_0^\infty \exp\left(-\frac{\tau^2}{4t}\right)u(x,\tau)\,d\tau.$$

$$v_t=Lv, x\in\mathbb{R}^n, t\in(0,1), \tag{17}$$

$$v(x,0)=f(x). \tag{18}$$

Denote

$$\mathcal{L}\varphi_1 = \overline{\varphi}_1(x,t) = v\mid_{S_1},\ \mathcal{L}\varphi_2 = \overline{\varphi}_2(x,t) = v\mid_{P_1}.$$

Let

$$\overline{\psi}_1(x,t) = \partial_\nu v\mid_{S_1}, \overline{\psi}_2(x,t) = \partial_{x_1} v\mid_{P_1}. \tag{19}$$

We obtain

$$\begin{aligned} \left\|\overline{\psi}_1\right\|_{C^{1+\alpha,\alpha/2}(\overline{S}_1)} &\leq C\left\|\overline{\varphi}_1\right\|_{C^{2+\alpha,1+\alpha/2}(\overline{S}_1)}, \\ \left\|\overline{\psi}_2\right\|_{C^{1+\alpha,\alpha/2}(\overline{P}_1)} &\leq C\left\|\overline{\varphi}_2\right\|_{C^{2+\alpha,1+\alpha/2}(\overline{P}_1)}, \end{aligned}$$

Conclusions

1. Therefore, each problem IP1, IP2 is now replaced with the Cauchy problem for the parabolic PDE with the lateral data.

2. To estimate $f(x)$, we now can use logarithmic stability estimates of initial conditions of parabolic PDEs: Klibanov, 2006 (finite domain) and Klibanov and Tikhonravov, 2007 (infinite domain).

3. Those estimates in turn were obtained via Carleman estimates.

The data after the Reznickaya transform.

Let

$$\|\varphi_1\|_{C^4(\overline{S}_T)} \leq \delta \exp\left(T^2/8\right), \forall T > 0$$

$$\|\varphi_2\|_{C^4(\overline{P}_T)} \leq \delta \exp\left(T^2/8\right), \forall T > 0,$$

where $\delta \in (0, 1)$ is a sufficiently small number. Then

$$\|\overline{\varphi}_1\|_{H^1(\overline{S}_1)} + \left\|\overline{\psi}_1\right\|_{L_2(\overline{S}_1)} \leq C\delta, \tag{20}$$

$$\|\overline{\varphi}_2\|_{H^1(G_1)} + \left\|\overline{\psi}_2\right\|_{L_2(G_1)} \leq C\delta, \tag{21}$$

where $G \subset P$ is an arbitrary bounded domain.

Theorem 1. IP1 (complete data collection). *Assume that the upper bound C_1 of the norm $\|\nabla f\|_{L_2(\Omega)}$ is given,*

$$\|\nabla f\|_{L_2(\Omega)} \leq C_1.$$

Then there exists a constant $M_1 > 0$ and a sufficiently small number $\delta_1 \in (0,1)$ such that if in (20) the number $\delta \in (0, \delta_1)$, then the following logarithmic stability estimate is valid

$$\|f\|_{L_2(\Omega)} \leq \frac{M_1 C_1}{\sqrt{\ln\left[(C_1\delta)^{-1}\right]}}.$$

Logarithmic stability

Theorem 2. IP2 (incomplete data collection). *Assume that the upper bound C_1 of the norm $\|f\|_{C^{2+\alpha}(\overline{\Omega})}$ be* given, i.e.

$$\|f\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_2.$$

Then there exists a constant $M_2 > 0$ and a sufficiently small number $\delta_2 \in (0,1)$ such that if the number δ in (21) is so small that $\delta \in (0, \delta_2)$, then

$$\|f\|_{L_2(\Omega)} \leq \frac{M_2 C_2}{\sqrt{\ln\left[(C_2\delta)^{-1}\right]}}.$$

Extension to the integral inequality

These results are extended via Carleman estimates to the case of integral inequalities like, e.g.

$$\iint\limits_{Q_1} (v_t - Lv)^2 \, dxdt \leq K, K = const. > 0. \tag{22}$$

We need (22) for the proof of convergence of the Quasi-Reversibility Method.

Quasi-Reversibility Method

Minimize the following Tikhonov functional

$$J_\gamma(v) = \|v_t - Lv\|^2_{L_2(Q_1)} + \gamma \|v\|^2_{H^4(Q_1)},$$

subject to the boundary conditions

$$v\mid_{S_1} = \overline{\varphi}_1, \partial_\nu v \mid_{S_1} = \overline{\psi}_1.$$

Assume the existence of the function $F \in H^{2,1}(Q_1)$ such that

$$F\mid_{S_1} = \overline{\varphi}_1, \partial_\nu F \mid_{S_1} = \overline{\psi}_1.$$

Let

$$w = v - F, \widetilde{F} = LF - F_t,$$

$$w \mid_{S_1} = \partial_\nu w \mid_{S_1} = 0.$$

Then

$$\overline{J}_\gamma (w) = \left\| w_t - Lw - \widetilde{F} \right\|^2_{L_2(Q_1)} + \gamma \|w\|^2_{H^4(Q_1)} \to \min .$$

Lemma 1. *For every function* $\widetilde{F} \in L_2(Q_1)$ *and every* $\gamma > 0$ *there exists unique minimizer* $w_\gamma = w_\gamma \left(\widetilde{F} \right) \in H^4_0(Q_1)$ *of the functional* $\overline{J}_\gamma$*. Furthermore, the following estimate holds*

$$\|w_\gamma\|_{H^4(Q_1)} \leq \frac{1}{\sqrt{2\gamma}} \left\| \widetilde{F} \right\|_{L_2(Q_1)} .$$

Quasi-Reversibility Method

Let

$$f_\gamma\left(x\right) = w_\gamma\left(x, 0\right)$$

Let w^* be the exact solution for the exact data F^*.
Let the error estimate be

$$\left\|\widetilde{F} - F^*\right\|_{L_2(Q_1)} \leq \omega.$$

Convergence Theorem. *Let* $\gamma = \gamma(\omega) = \omega \in (0,1)$*. Let the function* $w_{\gamma(\omega)} \in H_0^4(Q_1)$ *be the unique minimizer of the functional* $\overline{J}_\gamma$ *(Lemma 1). Let* $\|w^*\|_{H^4(Q_1)} \leq Y$*, where the upper estimate* $Y = const. \geq 1$ *is given. Then there exist a constant* $M_3 > 0$ *and a sufficiently small number* $\omega_0 \in (0,1)$ *such that if* ω *is so small that* $\left(Y^2+1\right)\omega \in (0,\omega_0)$*, then the following logarithmic convergence rate is valid*

$$\left\| f_{\gamma(\omega)} - w^*(x,0) \right\|_{L_2(\Omega)} \leq \frac{M_3 Y}{\sqrt{\ln\left(\omega^{-1}\right)}}.$$

Klibanov: arxiv 1303.0923v1 [math-ph] 5 Mar 2013

$$\Delta_x u + k^2 u - q(x) u = -\delta(x - x_0), x \in \mathbb{R}^3,$$

$$u(x, x_0, k) = O\left(\frac{1}{|x - x_0|}\right), |x| \to \infty,$$

$$\sum_{j=1}^{3} \frac{x_j - x_{j,0}}{|x - x_0|} \partial_{x_j} u(x, x_0, k) - iku(x, x_0, k) = o\left(\frac{1}{|x - x_0|}\right), |x| \to \infty.$$

$$\begin{aligned} q(x) &\in C^2(\mathbb{R}^3), q(x) = 0 \text{ for } x \in \mathbb{R}^3 \diagdown G, \\ q(x) &\geq 0. \end{aligned}$$

$$B_\varepsilon(y) = \{x : |x - y| < \varepsilon\}$$

Let $G_1 \subset \mathbb{R}^3$ be a convex bounded domain with its boundary $\partial_1 G = S \in C^1$. Let $\varepsilon \in (0,1)$ be a number. We assume that

$$\Omega \subset G_1 \subset G, dist\left(S, \partial G\right) > 2\varepsilon \text{ and } dist\left(S, \partial \Omega\right) > 2\varepsilon.$$

Inverse Problem 3 (IP3). Suppose that the function $q\left(x\right)$ is unknown for $x \in \Omega$ and known for $x \in \mathbb{R}^3 \diagdown \Omega$. Also, assume that the following function $f_1\left(x, x_0, k\right)$ is known

$$f_1\left(x, x_0, k\right) = \left|u\left(x, x_0, k\right)\right|, \forall x_0 \in S, \forall x \in B_\varepsilon\left(x_0\right), x \neq x_0, \forall k \in \left(a, b\right),$$

where $\left(a, b\right) \subset \mathbb{R}$ is an arbitrary interval. Determine the function $q\left(x\right)$ for $x \in \Omega$.

Theorem (uniqueness). *Consider IP3. Let two potentials* $q_1(x)$ *and* $q_2(x)$ *be such that* $q_1(x) = q_2(x) = q(x)$ *for* $x \in \mathbb{R}^3 \diagdown \Omega$. *Let* $u_1(x, x_0, k)$ *and* $u_2(x, x_0, k)$ *be corresponding solutions of the above forward problem Assume that*

$$|u_1(x, x_0, k)| = |u_2(x, x_0, k)|, \forall x_0 \in S, \forall x \in B_\varepsilon(x_0), x \neq x_0, \forall k \in (a, b).$$

Then $q_1(x) \equiv q_2(x)$.

- Applications in studies of reflectivity of neutrons.

- Three more phaseless inverse problems are considered in that preprint.

Previous Uniqueness Results

$$P(x) = \left| \iint\limits_{\Omega} h(\xi) e^{ix\xi} d\xi \right|^2, x \in \mathbb{R}^n, n = 1, 2.$$

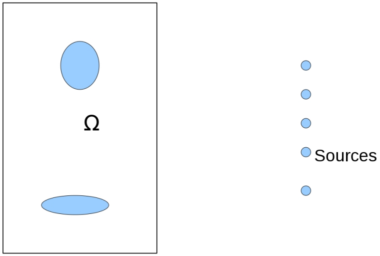
- $h(x) = A(x) \exp(i\varphi(x))$, where $A(x) = |h(x)|$
- Either $A(x)$ is known and $\varphi(x)$ is unknown, or vice versa.
- This is **Phase Retrieval Problem**. Klibanov 1985, 1987 (two papers), 2006.
- **Phaseless inverse scattering in 1-d**. Klibanov, 1989; Klibanov and Sacks, 1992. Survey: Klibanov, Sacks and Tikhonravov, 1995.

Diffusion Optical Tomography

$$\Delta u - a(\mathbf{x})\, u = -\delta(\mathbf{x} - \mathbf{x}_0)\,, \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^2, \quad (23)$$

$$\lim_{|\mathbf{x}| \to \infty} u(\mathbf{x}, \mathbf{x}_0) = 0. \quad (24)$$

- $a(\mathbf{x}) = 3(\mu_s' \mu_a)(\mathbf{x})$, where μ_s' is the reduced scattering coefficient and μ_a is the absorption coefficient.



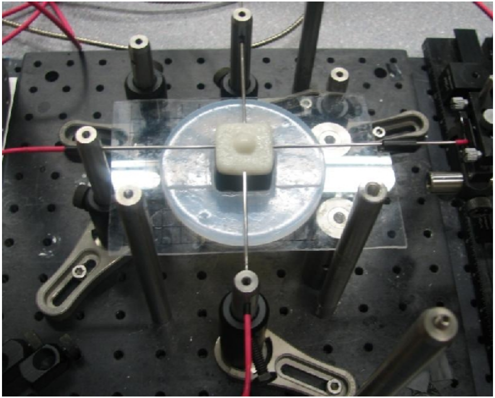
Inverse Problem. *Let $k = const. > 0$ be given. Suppose that in (23) the coefficient $a(\mathbf{x})$ satisfies the following conditions*

$$a \in C^1\left(\mathbb{R}^2\right), \; a(\mathbf{x}) \geq k^2 \text{ and } a(\mathbf{x}) = k^2 \text{ for } \mathbf{x} \in \mathbb{R}^2 \diagdown \Omega.$$

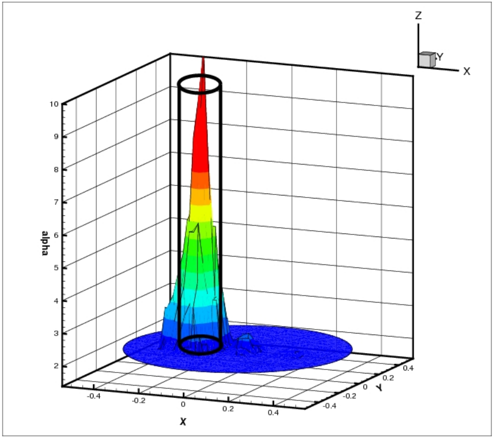
Let $L \subset \left(\mathbb{R}^2 \diagdown \overline{\Omega}\right)$ be a straight line and $\Gamma \subset L$ be an unbounded and connected subset of L. Determine the function $a(\mathbf{x})$ inside of the domain Ω, assuming that the constant k is given and also that the following function $\varphi(\mathbf{x}, \mathbf{x}_0)$ is given

$$u(\mathbf{x}, \mathbf{x}_0) = \varphi(\mathbf{x}, \mathbf{x}_0), \forall (\mathbf{x}, \mathbf{x}_0) \in \partial\Omega \times \Gamma.$$

Diffusion Optical Tomography



Diffusion Optical Tomography



Diffusion Optical Tomography

inclusion number	True contrast	Computed contrast	Relative error
1	2	2.11	5.6%
2	3	2.9	3.2%
3	4	4.22	5.7%
4	∞	6.69	unknown

Table: **Computed and correct inclusion/background contrasts and the relative errors**

Acknowledgments

This work was supported by the U.S. Army Research Laboratory and U.S. Army Research Office under the grant number W911NF-11-1-0399 and by the National Institutes of Health grant number 1R21NS052850-01A1.