Inverse spectral theory for surfaces of revolution

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We consider the Laplacian Δ_M on a rotationally symmetric manifolds M, see Fig. 1. Assume that $M = [0,1] \times Y$ is a cylindrical manifold with warped product metric

$$g = d^2x + r^2(x) g_0, \tag{1}$$

where the radius $r(x)$ is given by

$$r = e^{\frac{2}{m} Q}, \qquad Q(x) = \int_0^x q(t)dt, \quad x \in [0,1],$$
$$q \in W_1^0 = \left\{ q, q' \in L^2(0,1) \ ; \ q(0) = q(1) = 0 \right\}$$

Here (Y, g_0) is a compact m-dimensional Riemannian manifold (without boundary or with boundary). We call Y the transversal manifold. We need to mention that we work mostly with q, but that q determines the geometry (i.e., the function r and hence all derived quantities up to two integration constants).

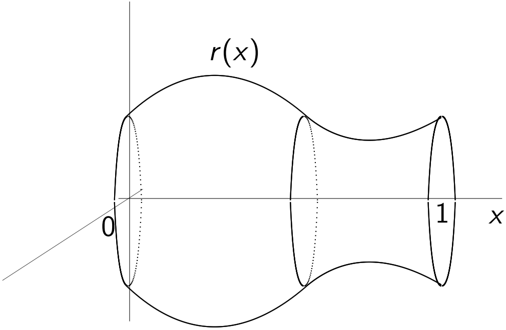

Figure: 1, The surface Y is a circle $\mathbb{S}^1$.

We discuss the manifold Y and the corresponding Laplacian Δ_Y:
Firstly, Y has not boundary. For example, we have a circle $Y = \mathbb{S}^1$, see Fig. 1. The operator Δ_Y has eigenvalues $E_1 = 0, E_2 = 1, E_3 = 1, E_4 = 2^2, ...$

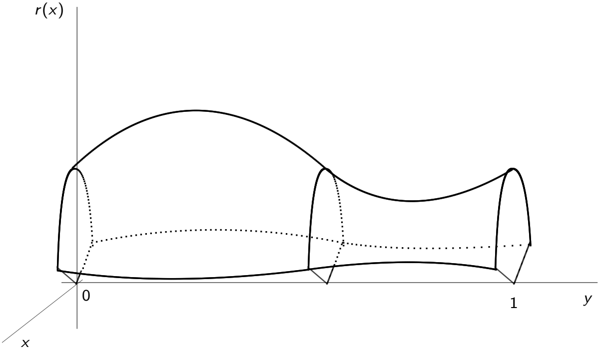

Figure: 2. The surface of revolution of an angle $\alpha < \pi$

Secondly, Y has a boundary. For example, we have a half of the circle, we can write $Y = [0,1]$, see Fig. 2. In the case of the Neumann boundary conditions the operator $-\Delta_Y$ has eigenvalues $E_1 = 0, E_2 = \pi^2, \ldots$. For the Dirichlet boundary conditions the operator $-\Delta_Y$ has eigenvalues $E_1 = \pi^2, E_2 = (2\pi)^2, \ldots$.

We consider the Laplacian Δ_M in $L^2(M)$ and for simplicity, below we consider only Dirichlet b.c. $f|_{\partial M} = 0$. In fact, we have results for more general case.

The Laplacian $-\Delta_Y$ on Y has discrete spectrum denoted by $0 \leqslant E_1 \leqslant E_2 \leqslant E_3 \leqslant ...$ with corresponding orthonormal family of eigenfunctions $\Psi_\nu, \nu \geqslant 1$ in $L^2(Y)$. The Laplacian Δ_M on M has the form

$$\Delta_M = \frac{1}{r^m}\partial_x r^m \partial_x + \frac{\Delta_Y}{r^2}.$$

Using the fact that $-\Delta_Y$ has discrete spectrum $E_\nu, \nu \geqslant 1$, we see that the Laplacian on (M, g) acting in $L^2(M)$ is unitarily equivalent to a direct sum of 1-dimensional operators Δ_ν:

$$-\Delta_M \backsimeq \oplus_{\nu=1}^{\infty} \Delta_\nu. \tag{2}$$

Here Δ_ν acts in the space $L^2([0,1], r^m(x)dx)$ and given by

$$\begin{gathered} \Delta_\nu f = -\frac{1}{r^m}(r^m f')' + \frac{E_\nu}{r^2}, \\ f = f(x),\ x \in [0,1], \quad f(0) = f(1) = 0, \end{gathered} \tag{3}$$

The problem is:

Determine $r(x)$ from the knowledge of the spectrum of Δ_ν.

The inverse spectral problem consists of the following parts:

i) Uniqueness. Prove that the spectral data uniquely determine the potential.

ii) Characterization. Give conditions for some data to be the spectral data of some potential.

iii) Reconstruction. Reconstruct the potential from spectral data.

Inverse problems for surfaces of revolution were discussed by Brüning-Heintze [84], Zelditch [98], Gurarie [95]. All these authors considered only Uniqueness.

We solve i)-iii). Our theorem is the first result about Characterization.

Introduce the space ℓ^2_α of real sequences $h = (h_n)_1^\infty$, equipped with the norm

$$\|h\|_\alpha^2 = \sum_{n \geqslant 1} n^{2\alpha} |h_n|^2, \qquad \alpha \in \mathbb{R}, \quad \textit{and} \quad \ell^2 = \ell^2_0.$$

We consider the Sturm-Liouville operator $\Delta_\nu, \nu \geqslant 1$ on the interval $[0, 1]$, with the Dirichlet boundary conditions:

$$\Delta_\nu f = -\frac{1}{r^m}(r^m f')' + \frac{E_\nu}{r^2} f, \qquad f(0) = f(1) = 0,$$

Denote by $\mu_n = \mu_n(q), n \geqslant 1$ the eigenvalues of Δ_ν. It is well known that all μ_n are simple and satisfy

$$\mu_n = \mu_n^0 + c_0 + \widetilde{\mu}_n, \quad \text{where} \quad (\widetilde{\mu}_n)_1^\infty \in \ell^2, \quad c_0 = \int_0^1 \left(q^2 + \frac{E_\nu}{r^2} \right) dt.$$

Here $(\pi n)^2$, $n \geqslant 1$ are the unperturbed eigenvalues for the case $r = 1$. Following Trubowitz we introduce the norming constants ("additional" spectral data)

$$\varkappa_n(q) = \log \left| \frac{r^{\frac{m}{2}}(1) f_n'(1, q)}{f_n'(0, q)} \right|, \qquad n \geqslant 1,$$

here f_n is the n-th eigenfunction, $f_n'(0, q) \neq 0$ and $f_n'(1, q) \neq 0$.

Theorem

Consider the inverse problem for Δ_ν for fixed $\nu \geqslant 1$ with Dirichlet b.c. The the mapping

$$\Psi : q \mapsto ((\widetilde{\mu}_n(q))_{n=1}^{\infty} ; (\varkappa_n(q))_{n=1}^{\infty})$$

is a real-analytic isomorphism between W_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where

$$\mathcal{M}_1 = \left\{ (h_n)_{n=1}^{\infty} \in \ell^2 : \mu_1^0 + h_1 < \mu_2^0 + h_2 < \ldots \right\} \subset \ell^2.$$

In particular, in the symmetric case the spectral mapping

$$\widetilde{\mu} : W_1^{0,odd} \to \mathcal{M}_1, \qquad \text{given by} \qquad q \to \widetilde{\mu} \tag{4}$$

is a real real analytic isomorphism between the Hilbert space $W_1^{0,odd} = \{q \in W_1^0 : q(x) = -q(1-x), \quad \forall\, x \in (0,1)\}$ and $\mathcal{M}_1$.

Few words about Proof. Consider the Sturm-Liouville operator Δ_ν in $L^2((0,1); r^m dx), r^{\frac{m}{2}} = e^Q, \quad Q = \int_0^x q(t)dt$, given by

$$\Delta_\nu f = -\frac{1}{r^m}(r^m f')' + \frac{E_\nu}{r^2} f, \qquad f(0) = f(1) = 0 \qquad r^{\frac{m}{2}} = e^Q,$$

We define the simple unitary transformation $\mathscr{U}$ by

$$\mathscr{U} : L^2([0,1], r^m dx) \to L^2([0,1], dx), \qquad y = \mathscr{U} f = r^{\frac{m}{2}} f.$$

We transform Δ_ν into the Schrödinger operator S_p in $L^2(0,1)$ by

$$\mathscr{U}(\Delta_\nu)\mathscr{U}^{-1} = -\frac{1}{r^{\frac{m}{2}}}\partial_x r^m \partial_x \frac{1}{r^{\frac{m}{2}}} + \frac{E_\nu}{r^2} = S_p + c_0, \quad S_p y = -y'' + py,$$

with b.c. $y(0) = y(1) = 0$, where

$$p = P(q) = q' + q^2 + \frac{E_\nu}{r^2} - c_0, \qquad c_0 = \int_0^1 \left(q' + q^2 + \frac{E_\nu}{r^2} \right) dx. \quad (5)$$

Inverse problems for the operator $S_p = -\frac{d^2}{dx^2} + p$ with different b.c. are well understood: Trubowitz+ coauthors for Dirichlet b.c., and Chelkak-Korotyaev for other b.c.. Thus we have the well understood mapping $p \to \{$ eigenvalues + norming constants $\}$ and we need to know the image of $p \in P(W_1^0) = ???$

Theorem

The mapping $P : \mathscr{W}_1^0 \to \mathscr{H}_0 = \{p \in L^2(0,1), \int_0^1 p dx = 0\}$ *given by* $p = P(q) = q' + q^2 + \frac{E_\nu}{r^2} - c_0, c_0 = \int_0^1 \left(q' + q^2 + \frac{E_\nu}{r^2}\right) dx$ *is a real analytic isomorphism between the Hilbert spaces* $\mathscr{W}_1^0$ *and* $\mathscr{H}_0$*.*

In order to prove this Theorem we use the "direct approach" of Kargaev-Korotyaev [97] based on nonlinear functional analysis. Recall the basic theorem of this approach.

Theorem

Let H, H_1 *be real separable Hilbert spaces equipped with norms* $\|\cdot\|, \|\cdot\|_1$*. Suppose that the map* $f : H \to H_1$ *satisfies:*
i) f *is real analytic and the operator* $\frac{d}{dq} f$ *has an inverse for* $\forall q \in H$*,*
ii) there exists an increasing function $\eta : [0, \infty) \to [0, \infty), \eta(0) = 0$*, such that* $\|q\| \leqslant \eta(\|f(q)\|_1)$ *for all* $q \in H$*,*
iii) there exists a linear isomorphism $f_0 : H \to H_1$ *such that the mapping* $f - f_0 : H \to H_1$ *is compact.*
Then f *is a real analytic isomorphism between* H *and* H_1*.*