Acceleration of the modified alternating algorithm by the conjugate gradient method for the Cauchy problem for the Helmholtz equation

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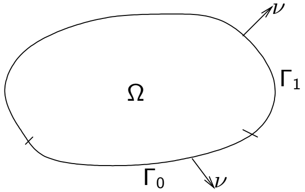
Outline

- Cauchy problem for the Helmholtz equation
- Alternating iterative algorithm
- Modified alternating algorithm
- Conjugate gradient method

Formulation of the Cauchy Problem for the Helmholtz equation

- Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary Γ.
- The boundary Γ is divided into two parts Γ_0 and Γ_1.



- Consider the Cauchy problem for the Helmholtz equation:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \Gamma_0, \\ \partial_\nu u = g & \text{on} \quad \Gamma_0, \end{cases}$$

  where k is the wave number.
- The problem is ill–posed.
- Applications: characterization of sound sources (Langrenne and Garcia: 2011), . . .

Alternating algorithm

Following

V.A. Kozlov and V.G. Maz'ya, *On iterative procedures for solving ill–posed boundary value problems that preserve differential equations*, Algebra i Analiz, 192 (1989), pp. 1207–1228,(in Russian).

the alternating algorithm may be described in the following way:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma_0, \\ \partial_\nu u = \eta & \text{on } \Gamma_1, \end{cases} \quad (1) \qquad \begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \Gamma_0, \\ u = \phi & \text{on } \Gamma_1, \end{cases} \quad (2)$$

1. The first approximation u_0 to the solution u is obtained by solving (1), where η is an arbitrary initial approximation of the normal derivative on Γ_1.
2. Having constructed u_{2n}, we find u_{2n+1} by solving (2) with $\phi = u_{2n}$ on Γ_1.
3. We then obtain u_{2n+2} by solving the problem (1) with $\eta = \partial_\nu u_{2n+1}$ on Γ_1.

Previous works

- V.A. Kozlov, V.G. Maz'ya and A.V. Fomin, *An iterative method for solving the Cauchy problem for elliptic equations*, Comput. Maths. Math. Phys., 31 (1991), no. 1, 46–52.
- D. Lesnic, L. Elliot and D.B. Ingham, *An alternating BEM for solving numerically the Cauchy problems for the Laplace equation*, Engineering Analysis with Boundary Elements, 20 (1997), no. 2, pp. 123–133.
- S. Avdonin, V. Kozlov, D. Maxwell and M. Truffer, *Iterative methods for solving a nonlinear boundary inverse problem in glaciology*, J. Inv. Ill-Posed Problems, 17 (2009), pp. 239–258.
- R. Chapko and B.T. Johansson, *An alternating potential–based approach to the Cauchy problem for the Laplace equation in a planar domain with a cut*, Comp.Meth. Appl. Math., 8 (2008), no. 4, pp. 315–335.
- G. Bastay, T. Johansson, V.A. Kozlov and D. Lesnic, *An alternating method for the stationary Stokes system*, Z. Angew. Math. Mech., 86 (2006), no. 4, pp. 268–280.
- L. Marin, L. Elliott, P.J. Heggs, D.B. Ingham, D. Lesnic and X. Wen, *An alternating iterative algorithm for the Cauchy problem associated to the Helmholtz equation*, Comput. Meth. Appl. Mech. Eng., 192 (2003), pp. 709–722.

Nonconvergence of the original algorithm for the Cauchy problem for the Helmholtz equation

- Consider the Cauchy problem for the Helmholtz equation in a rectangle $[0, a] \times [0, b]$:

$$\begin{cases} \Delta u(x,y) + k^2 u(x,y) = 0, & 0 < x < a, \quad 0 < y < b, \\ u(x,0) = f(x), & 0 \le x \le a, \\ u_y(x,0) = g(x), & 0 \le x \le a, \\ u(0,y) = u(a,y) = 0, & 0 \le y \le b. \end{cases}$$

- This problem is ill–posed.
- The algorithm diverges for

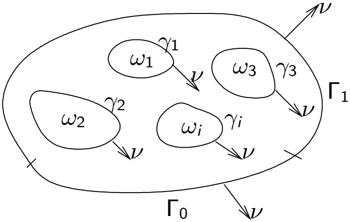
$$k^2 \ge \pi^2 (a^{-2} + (4b)^{-2})$$

Choice of the interior boundary

B.T. Johansson and V.A. Kozlov, *An alternating method for Helmholtz–type operators in non-homogeneous medium*, IMA Journal of Applied Mathematics, 74 (2009), pp. 62–73.

F. Berntsson, V.A. Kozlov, L. Mpinganzima and B.O. Turesson, An alternating iterative procedure for the Cauchy problem for the Helmholtz equation, accepted by the Journal of Inverse Problems in Science and Engineering (Proceedings).

- Introduce open subsets ω_i, $i = 1, \ldots, n$ inside Ω with boundaries γ_i, $i = 1, \ldots, n$.
- We assume that every ω_i is a Lipschitz domain.



- $\Omega_1 = \cup_{i=1}^{n} \omega_i$ with Lipschitz boundary $\gamma = \cup_{i=1}^{n} \gamma_i$ and $\Omega_2 = \Omega \backslash (\Omega_1 \cup \gamma)$.

Assumption: For all nonzer u,

$$\int_\Omega \left(|\nabla u|^2 - k^2 u^2\right)\, dx + \mu \int_\gamma u^2 \, dS > 0,$$

for $u \in H^1(\Omega)$ such that $u \neq 0$.

Sufficient condition for the positivity

Theorem

Let

$$\Lambda_\mu = \min_{\substack{u \in H^1(\Omega) \\ \|u\|_2 = 1}} \int_\Omega |\nabla u|^2 \, dx + \mu \int_\gamma u^2 \, dS,$$

and

$$\Lambda = \min_{\substack{u \in H^1(\Omega), u|_\gamma = 0 \\ \|u\|_2 = 1}} \int_\Omega |\nabla u|^2 \, dx.$$

Then there exists a positive constant C such that

$$\Lambda - \Lambda_\mu \le \frac{C(\Lambda)^{3/2}}{\mu^{1/2}}.$$

Corollary

Corollary

If Λ is positive, then

$$\int_\Omega \left(|\nabla u|^2 - k^2 u^2\right)\, dx + \mu \int_\gamma u^2 \, dS > 0, \quad \textit{for all} \quad u, \quad u \neq 0 \quad \textit{on} \quad \gamma.$$

for sufficiently large μ.

Modified alternating iterative algorithm for the Cauchy problem for the Helmholtz equation

The modified algorithm will consist of solving the following well–posed problems alternatively:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega\backslash\gamma, \\ u = f & \text{on } \Gamma_0, \\ \partial_\nu u = \eta & \text{on } \Gamma_1, \\ [\partial_\nu u] + \mu u = \xi & \text{on } \gamma, \\ [u] = 0 & \text{on } \gamma, \end{cases} \quad (3) \qquad \begin{cases} \Delta v + k^2 v = 0 & \text{in } \Omega\backslash\gamma, \\ \partial_\nu v = g & \text{on } \Gamma_0, \\ v = \phi & \text{on } \Gamma_1, \\ v = \varphi & \text{on } \gamma. \end{cases} \quad (4)$$

1. The first approximation u_0 to the solution u is obtained by solving (3), where η is an arbitrary initial approximation of the normal derivative on Γ_1 and ξ is an arbitrary approximation of $[\partial_\nu u] + \mu u$ on γ.
2. Having constructed u_{2n}, we find u_{2n+1} by solving (4) with $\phi = u_{2n}$ on Γ_1 and $\varphi = u_{2n}$ on γ.
3. We then obtain u_{2n+2} by solving the problem (3) with $\eta = \partial_\nu u_{2n+1}$ on Γ_1 and $\xi = [\partial_\nu u_{2n+1}] + \mu u_{2n+1}$ on γ.

Convergence of the modified alternating iterative algorithm

Theorem

Let $f \in H^{1/2}(\Gamma_0)$ and $g \in H^{1/2}(\Gamma_0)^*$, and let $u \in H^1(\Omega)$ be the solution to the Cauchy problem for the Helmholtz equation given above. Then, for every $\eta \in H^{1/2}(\Gamma_1)^*$ and every $\xi \in H^{1/2}(\gamma)^*$, the sequence $(u_n)_{n=0}^{\infty}$ obtained from the modified alternating algorithm converges to u in $H^1(\Omega)$.

Modified algorithm continued

Given $\eta \in H^{1/2}(\Gamma_1)^*$ and $\xi \in H^{1/2}(\gamma)^*$, let us define

$$B(\eta, \xi) = (\partial_\nu v\big|_{\Gamma_1}, [\partial_\nu v] + \mu v\big|_\gamma).$$

We find that

$$(\eta_{k+1}, \xi_{k+1}) = B(\eta_k, \xi_k).$$

- Consider the following problem

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega\backslash\gamma, \\ u = 0 & \text{on } \Gamma_0, \\ \partial_\nu u = \eta & \text{on } \Gamma_1, \\ [\partial_\nu u] + \mu u = \xi & \text{on } \gamma, \\ [u] = 0 & \text{on } \gamma, \end{cases}$$

- Introduce a linear operator $N : H^{1/2}(\Gamma_1)^* \times H^{1/2}(\gamma)^* \longrightarrow H^{1/2}(\Gamma_0)^*$ by

$$N(\eta, \xi) = \partial_\nu u\big|_{\Gamma_0},$$

where $\eta \in H^{1/2}(\Gamma_1)^*$, $\xi \in H^{1/2}(\gamma)^*$.

- If $u \in H^1(\Omega)$ solves the Cauchy problem for the Helmholtz equation with $f = 0$ on Γ_0, the problem can then be formulated as

$$N(\eta, \xi) = g.$$

Adjoint operator N^*

Lemma

Let $\zeta \in H^{1/2}(\Gamma_0)^*$*, and let v solves the*

$$\begin{cases} \Delta w + k^2 w = 0 & in\, \Omega \backslash \gamma, \\ \partial_\nu w = \zeta & on\, \Gamma_0, \\ w = 0 & on\, \Gamma_1, \\ w = 0 & on\, \gamma. \end{cases}$$

Then $N^*(\zeta) = (\partial_\nu w\big|_{\Gamma_1}, [\partial_\nu w] + \mu w\big|_\gamma)$.

Landweber method

- Consider the following functional

$$J(\eta, \xi) = \|g - N(\eta, \xi)\|_{H^{1/2}(\Gamma_0)^*}$$

- Let us define

$$L_N(\eta, \xi) = (\eta, \xi) + \alpha N^*(g - N(\eta, \xi)),$$

  where α is a fixed constant chosen so that $0 < \alpha < \|N\|^{-2}$.
- The Landweber method produces iterates

$$(\eta_{k+1}, \xi_{k+1}) = L_N(\eta_k, \xi_k).$$

Modified algorithm and Landweber method

Theorem

For any $\eta \in H^{1/2}(\Gamma_1)^*$ and $\xi \in H^{1/2}(\gamma)$, the iterates produced by the Landweber method and the modified alternating algorithm are identical, i.e.,

$$L_N(\eta, \xi) = B(\eta, \xi). \tag{5}$$

Conjugate gradient method

The conjugate gradient method for the problem is as follows

1. Choose initial $\eta_0 \in H^{1/2}(\Gamma_1)^*$ and $\xi_0 \in H^{1/2}(\gamma)^*$.
   Denote $\chi_0 = (\eta_0, \xi_0)$ and $(H^{1/2})^* = H^{1/2}(\Gamma_1)^* \times H^{1/2}(\gamma)^*$.
2. $d_0 = g - N(\chi_0)$;
3. $p_1 = s_0 = N^*(d_0)$;
4. for $k = 1, 2, \ldots,$ unless $s_{k-1} = 0$, compute
5. $q_k = N(\chi_k)$;
6. $\alpha_k = \|s_{k-1}\|_{(H^{1/2})^*} / \|q_k\|_{H^{1/2}(\Gamma_0)^*}$;
7. $\chi_k = \chi_{k-1} + \alpha_k p_k$;
8. $d_k = d_{k-1} - \alpha_k q_k$;
9. $s_k = N * (d_k)$;
10. $\alpha_k = \|s_k\|_{(H^{1/2})^*} / \|s_{k-1}\|_{(H^{1/2})^*}$;
11. $p_{k+1} = s_k + \beta_k p_k$.

Numerical experiments

- The domain is the rectangle $\Omega = (0,1) \times (0,L)$.
- We put $\Gamma_0 = (0,1) \times \{0\}$ and $\Gamma_1 = (0,1) \times \{L\}$.
- We choose $L = 0.2$, the computational grid $N = 401$, and $M = 81$ and the following exact data:

$$u(x,0) = \left(3\sin \pi x + \frac{\sin 3\pi x}{19} + 9\exp(-30(x-L)^2)\right) x^2(1-x)^2,$$

and

$$u(x,L) = 2\left(8\sin \pi x + \frac{\sin 3\pi x}{17} + 20\exp(-50(x-L)^2)\right) x^2(1-x)^2.$$

Numerical experiments

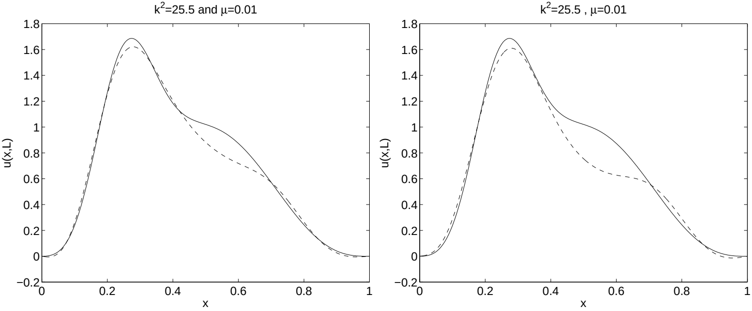


Figure 1 : Modified algorithm (left) after 1500 iterations and the conjugate gradient method (right) after 20 iterations.

THANK YOU FOR YOUR ATTENTION.