Underwater topography “invisible” for surface waves at given frequencies

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The invisibility.

Approaches:

- smartly designed composite materials + special shapes (STEALTH technology)
- local deformation of space variables
  (HARRY POTTER'S cloack)
- *Greenleaf A., Kurylev Ya., Lassas M. Uhlmann G.* Invisibility an inverse problems. Bull. Amer. Math. Soc. 2009. V. 46.
- *Greenleaf A., Kurylev Ya., Lassas M. Uhlmann G.*
  Approximate quantum and acoustic cloacking.
  J. Spectr. Theory. 2011. V. 1.

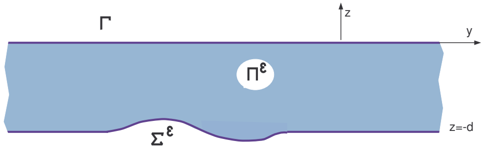
Our approach:

- No changes in the differential equations and the boundary conditions but only design of obstacle's shape.
- The main difference is that we consider WAVEGUIDES and thus deal with a FINITE number of propagative waves.

The linear theory of water-waves.

The Steklov spectral problem:

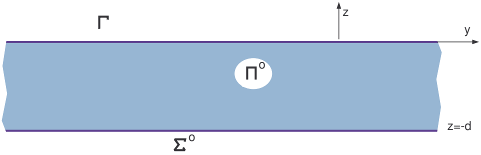
- $$-\Delta\varphi^\varepsilon + k^2\varphi^\varepsilon = 0 \quad \text{in} \quad \Pi^\varepsilon,$$
- $$\partial_z\varphi^\varepsilon = \lambda\varphi^\varepsilon \quad \text{on} \quad \Gamma,$$
- $$\partial_\nu\varphi^\varepsilon = 0 \quad \text{on} \quad \Sigma^\varepsilon,$$
- φ^ε is the velocity potential and $\lambda^\varepsilon = g^{-1}\omega^2$ the spectral parameter with a frequency $\omega > 0$ and the acceleration $g > 0$ due to gravity,
- the superscript $\varepsilon > 0$ indicates the size of the perturbation of the bottom (a warp) $\Sigma^\varepsilon = \{(y,z) : y \in \mathbb{R}, z = -d + \varepsilon h(y)\}$.



Propagative waves.

The continuous spectrum $[\lambda_\dagger, +\infty)$.

- The cutoff point $\lambda_\dagger = \lambda(k) \geq 0$ satisfies
$$\lambda(m) = m\frac{1 - e^{-2md}}{1 + e^{-2md}} \text{ with } m \geq 0.$$
- If the bottom is flat, i.e., $h = 0$, then, for any $l \geq 0$, there exists in the straight channel two propagative waves
$$w^{\pm}(y,z) = e^{\pm ily}(e^{mz} + e^{-m(z+2d)})$$
where $m = \sqrt{k^2 + l^2}$.
- According to the Sommerfeld principle the wave w^+ travels from $-\infty$ to $+\infty$.



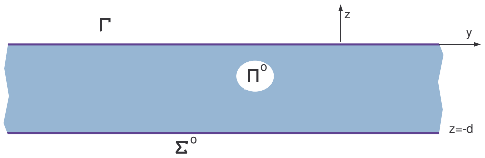
The transmission and reflection coefficients.

The straight channel Π^0.

- The wave $w^+(y,z) = e^{+ily}(e^{mz} + e^{-m(z+2d)})$, of course, travels from $-\infty$ to $+\infty$ without any perturbation.

The perturbed channel Π^ε.

- The scattered wave:
  $u^\varepsilon(y,z) = \chi_-(y)w^+(z) + \sum_\pm \chi_\pm(y)s_\pm^\varepsilon w^\pm(z) + \widetilde{u}^\varepsilon(y,z)$,
- where the reflection s_-^ε and transmission s_+^ε coefficients satisfy $|s_-^\varepsilon|^2 + |s_+^\varepsilon|^2 = 1$, $\widetilde{u}^\varepsilon$ decays exponentially, $\chi_\pm$ are cut-off functions near $y = \pm\infty$.



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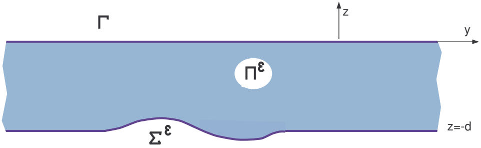
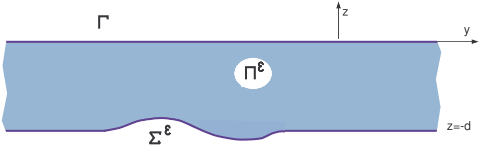
- where the reflection s^ε_- and transmission s^ε_+ coefficients satisfy $|s^\varepsilon_-|^2 + |s^\varepsilon_+|^2 = 1$, $\widetilde{u}^\varepsilon$ decays exponentially, $\chi_\pm$ are cut-off functions near $y = \pm\infty$.



Invisibility.

The perturbed channel Π^ε.

- The scattered wave:
  $$u^\varepsilon(y,z) = \chi_-(y)w^+(z) + \sum_\pm \chi_\pm(y)s^\varepsilon_\pm w^\pm(z) + \widetilde{u}^\varepsilon(y,z).$$
- We fix some $l > 0$ and the frequency $\omega_l = \sqrt{g\lambda(k,l)^{1/2}}$.
- The perturbation (obstacle) is "**invisible**" with the reflection coefficient $s^\varepsilon_- = 0$ and transmission coefficient $s^\varepsilon_+ = 1$.
- The perturbation (obstacle) is **non-reflective** with the reflection coefficient $s^\varepsilon_- = 0$ (and, hence, the transmission coefficient $s^\varepsilon_+ = e^{i\psi_\varepsilon}$).



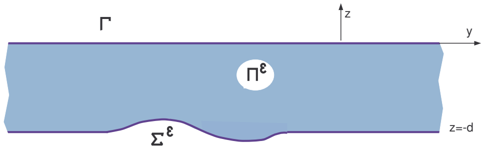
Invisibility.

The "invisible" local perturbation (a warp) at a given frequency.

- One needs to find out a **smooth** profile $h(y)$ of the slightly sloped bottom $\Sigma^\varepsilon = \{(y,z) : y \in \mathbb{R}, z = -d + \varepsilon h(y)\}$ (**with** $\mathrm{supp} h \subset (-L,+L)$, $L > 0$) such that

$$s_-^\varepsilon = 0 \quad \text{and} \quad s_+^\varepsilon = 1$$

in the solution
$$u^\varepsilon(y,z) = \chi_-(y)w^+(z) + \sum_\pm \chi_\pm(y)s_\pm^\varepsilon w^\pm(z) + \widetilde{u}^\varepsilon(y,z).$$



Invisibility.

The “invisible” perturbation.

- We accept the asymptotic ansätze
  $u^\varepsilon(y,z) = w^+(y,z) + \varepsilon u'(y,z) + \dots,$
  $s^\varepsilon_\pm = s^0_\pm + \varepsilon s'_\pm + \dots,$
  where $u^0(y,z) = w^+(y,z)$ and $s^0_+ = 1$, $s^0_- = 0$
- .
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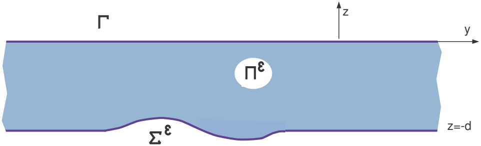
- .



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- and then rectify the bottom
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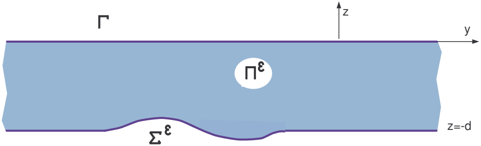

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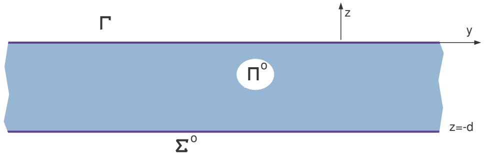
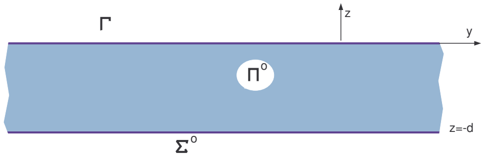
  where $u^0(y,z) = w^+(y,z)$ and $s^0_+ = 1$, $s^0_- = 0$
- and then rectify the bottom
- taking into account that
  $\partial_\nu = (1+\varepsilon^2|\partial_y h(y)|^2)^{-1/2}(-\partial_z + \varepsilon \partial_y h(y)).$



Asymptotic analysis.

The boundary condition at the rectified bottom.

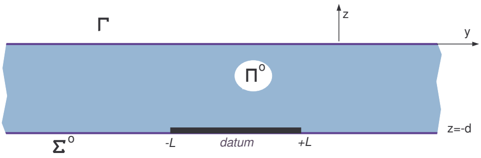
- $\partial_\nu u^\varepsilon(y,-d+\varepsilon h(y)) = -\partial_z u^0(y,-d+\varepsilon h(y)) +$
  $+\varepsilon\partial_y h(y)\partial_z u^0(y,-d+\varepsilon h(y)) - \varepsilon\partial_z u'(y,-d+\varepsilon h(y)) + \cdots =$
  $= -\partial_z u^0(y,-d) - \varepsilon h(s)\partial_z^2 u^0(y,-d) +$
  $+\varepsilon\partial_y h(y)\partial_z u^0(y,-d) - \varepsilon\partial_z u'(y,-d) + \ldots.$
- The Helmholtz equation provides
  $\partial_z^2 u^0(y,-d) = -\partial_y^2 u^0(y,-d) + k^2 u^0(y,-d)$
- and therefore $\partial_\nu u^\varepsilon(y,-d+\varepsilon h(y)) =$
  $= \varepsilon\big(-\partial_z u'(y,-d) + \partial_y h(y)\partial_z u^0(y,-d) +$
  $+h(y)\partial_y^2 u^0(y,-d) - h(y)k^2 u^0(y,-d)\big) + \ldots$



Asymptotic analysis.

The problem for the correction term.

- Thus, the function u' satisfies
  $-\Delta u'(y,z) + k^2 u'(y,z) = 0, \ (y,z) \in \Pi^0,$
  $\partial_z u'(y,0) = \lambda u'(y,0), \ y \in \mathbb{R},$
  $-\partial_z u'(y,-d) = -\partial_y(h(y)\partial_y u^0(y,-d)) + k^2 h(y) u^0(y,-d).$
- There exists a unique solution such that
- $u'(y,z) = \sum_{\pm} \chi_{\pm}(y) s'_{\pm} w^{\pm}(z) + \widetilde{u}\,'(y,z)$
  with some coefficients $s'_{\pm}$.



Asymptotic analysis.

Formulas for the coefficients.

- The correction term
  $u'(y,z) = \sum_{\pm} \chi_{\pm}(y) s'_{\pm} w^{\pm}(z) + \widetilde{u}'(y,z).$
- Insert $u'(y,z)$ and $w^{\pm}(y,z)$ into the Green formula in the rectangle $(-R,R)\times(-d,0)$ and send R to ∞.
- As a result we obtain:
  $s'_+ = 4iN^{-1}(k^2+l^2)\int_{-L}^{L} h(y)dy,$
  $s'_- = 4iN^{-1}(k^2-l^2)\int_{-L}^{L} e^{2ily}h(y)dy$
  with a certain $N>0$.

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Asymptotic analysis.

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  $$s'_{+} = 4iN^{-1}(k^2+l^2)\int_{-L}^{L} h(y)dy,$$
  $$s'_{-} = 4iN^{-1}(k^2-l^2)\int_{-L}^{L} e^{2ily}h(y)dy$$
- Thus imposing three orthogonality conditions
  $$\int_{-L}^{L} h(y)dy = 0, \quad \int_{-L}^{L} e^{2ily}h(y)dy = 0 \in \mathbb{C}$$
- provides $s'_{\pm} = 0$,
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- Thus imposing three orthogonality conditions
  $$\int_{-L}^{L} h(y)dy = 0, \quad \int_{-L}^{L} e^{2ily} h(y)dy = 0 \in \mathbb{C}$$
- provides $s'_\pm = 0$,
- however, we still cannot achieve $s^\varepsilon_\pm = 0$
  because the lower-order perturbation $\varepsilon^2 \widetilde{s}^\varepsilon_\pm$.

Searching for a proper profile.

Complicating the form of the profile.

- To find out an "invisible" warp we employ an idea and techniques of the **enforced stability** of embedded eigenvalues, see, e.g.,

  Nazarov S.A. Trapped waves in a cranked waveguide with hard walls **Acoustical Physics** 2011. V. 57 (6). P. 764-771,

  Nazarov S.A. Asymptotic expansions of eigenvalues in the continuous spectrum of a regularly perturbed quantum waveguide **Theoretical and mathematical physics** 2011. V. 167 (2). P. 606–627.

- We linearize the equations $s_+^\varepsilon = 1$ and $s_-^\varepsilon = 0$ around the asymptotic solution.

Searching for a proper profile.

Complicating the form of the profile.

- To find out an "invisible" warp we employ the techniques of the **enforced stability** of embedded eigenvalues,
  namely we impose the decomposition of the profile
  $$h(y) = h_0(y) + \sum_{j=1}^{3} \tau_j(\varepsilon) h_j(y)$$
  where $\tau = (\tau_1, \tau_2, \tau_3)$ is the vector of new small parameters and $h_q \in C_c^4(-L, +L)$ ...
- We linearize the equations $s_+^\varepsilon = 1$ and $s_-^\varepsilon = 0$ around the asymptotic solution.

Complicating the form of the profile.

- $h(y) = h_0(y) + \sum_{j=1}^{3} \tau_j(\varepsilon) h_j(y)$
  where $\tau = (\tau_1, \tau_2, \tau_3)$ is the vector of new small parameters
  and $h_q \in C_c^4(-L, +L)$ subject
  to the normalization and orthogonality conditions
  $$\int_{-L}^{L} R_k(y) h_0(y) dy = 0, \quad \int_{-L}^{L} R_k(y) h_j(y) dy = \delta_{j,k},$$
  where $j, k = 1, 2, 3$ and
  $$R_1(y) = 1, \quad R_2(y) = \cos(2ly), \quad R_3(y) = \sin(2ly)$$
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  where $j, k = 1, 2, 3$ and
  $$R_1(y) = 1, \quad R_2(y) = \cos(2ly), \quad R_3(y) = \sin(2ly)$$
- notice that $e^{2ily} = \cos(2ly) + i\sin(2ly)$.

The equations to determine the desired profile.

Transcendental equations.

- We have $s^\varepsilon_\pm = s^0_\pm + \varepsilon s'_\pm + \varepsilon^2 \widetilde{s}^\varepsilon_\pm$,

$$s'_+ = 4iN^{-1}(k^2+l^2)\int_{-L}^{L} h(y)dy,$$

$$s'_- = 4iN^{-1}(k^2-l^2)\int_{-L}^{L} e^{2ily} h(y)dy$$

- Then three equations $\mathrm{Im} s^\varepsilon_+ = 0$ and $s^\varepsilon_- = 0 \in \mathbb{C}$ under the condition $k \neq l$ reduce to the abstract equation

$$\tau = T^\varepsilon(\tau) \quad \text{in} \quad \mathbb{R}^3$$

- where $T^\varepsilon_1(\tau) = -\frac{\varepsilon}{4}\frac{1}{k^2+l^2} N \mathrm{Im}(\widetilde{s}^\varepsilon_+)$, $\quad T^\varepsilon_2(\tau) =$

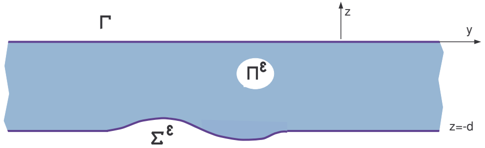
$$= -\frac{\varepsilon}{4}\frac{1}{k^2-l^2} N \mathrm{Im}(\widetilde{s}^\varepsilon_+), \quad T^\varepsilon_3(\tau) = \frac{\varepsilon}{4}\frac{1}{k^2-l^2} N \mathrm{Re}(\widetilde{s}^\varepsilon_+).$$

- The most important point is: for a small ε
the operator T^ε in $\mathbb{R}^3$ is contractive in a small ball !

The properties of the operator T^ε.

Estimates for the remainder.

- Rectifying the boundary: the coordinate change
$$(y, z) \mapsto (y^\varepsilon, z^\varepsilon),$$
- namely, we make the local shift near the warp
$$y^\varepsilon = y, \quad z^\varepsilon = z - \varepsilon h(y)$$
which slightly and **analytically in** ε and τ
perturbs the operators and
glue it with identity outside the vicinity of the warp.
- Profits:
the estimate $|\widetilde{s}_\pm^\varepsilon| \leq c$ in the remainder $\widetilde{s}_\pm^\varepsilon$
and the analytic dependence of $s_\pm^\varepsilon$ on τ.



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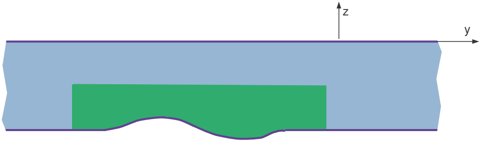

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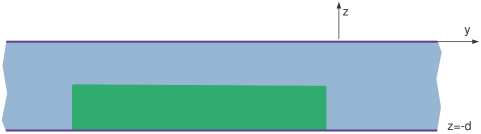


Conclusive remarks.

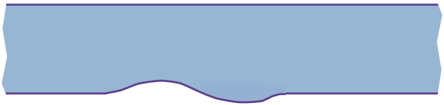
About the warp.

- The solution $\tau = \tau(\varepsilon)$ exists and unique in the ball $\mathbb{B}^3_{\varepsilon\rho}$ with some $\rho > 0$.
- Since $|\tau(\varepsilon)| \leq \varepsilon\rho$, we have

$$\varepsilon h(y) = \varepsilon h_0(y) + \sum_{j=1}^{3} \tau_j(\varepsilon) h_j(y) = \varepsilon h_0(y) + O(\varepsilon^2),$$

i.e., h_0 is the main term under three conditions only.

- In view of the condition $\int_{-L}^{L} h_0(y)dy = 0 \quad (*)$ the increment of the volume due to the warp is $O(\varepsilon^2)$.



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- In view of the condition $\int_{-L}^{L} h_0(y)dy = 0 \quad (*)$
  the increment of the volume due to the warp is $O(\varepsilon^2)$.
- Aiming to make the warp non-reflecting only,
  we may omit $(*)$ and make the volume increment $\geq c\varepsilon$.
- .
  .
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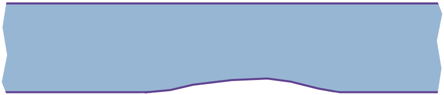

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  thus we cannot consider more complicated shapes yet.
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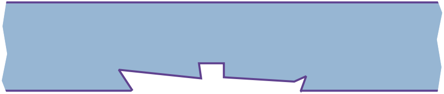

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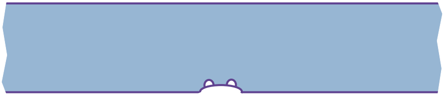

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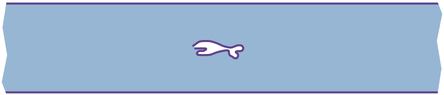
- The proof is based on the smoothness assumption on h_p, thus we cannot consider more complicated shapes yet.
- We also do not know yet about "invisible" submerged objects.



Thanks a lot for attention !