A uniform reconstruction formula in integral geometry

Victor Palamodov

Tel Aviv University

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Given a function $f$ (or a differential form) defined in a Riemannian manifold $(X, g)$ the problem is to recover $f$ from data of integrals

$$Rf(\sigma) = \int_{Z(\sigma)} f \, dg S$$

along a family of varieties $Z(\sigma) \subset X, \sigma \in \Sigma$ against a Riemannian area density $dg S$. The set $\Sigma$ is called acquisition geometry.
Classical reconstructions

- The classical reconstruction formulas (Radon-John) hold for family of hyperplanes in $\mathbb{R}^n$ and are FBP type. In the case $n = 2$

$$f(x) = -\frac{1}{4\pi^2} \int_0^{2\pi} \left( \int_{\mathbb{R}} \frac{g_p'(p, \omega)}{\langle \omega, x \rangle - p} \, dp \right) \, d\omega$$

filtration is composition of the Hilbert transform and first derivative. H.Lorentz’s reconstruction for the case $n = 3$

$$f(x) = -\frac{1}{8\pi^2} \int_{S^2} g_p''(\langle \omega, x \rangle, \omega) \, d\Omega$$

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There are several reconstruction formulas that look similar.
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A general approach

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- $Z$ be a smooth closed hypersurface in $X \times \Sigma$ and $p : Z \to X$, $\pi : Z \to \Sigma$ be natural projections,
- $Z$ is zero level set for a generating function $\Phi : X \times \Sigma \to \mathbb{R}$ such that $d_x \Phi \neq 0$ on $Z$. 
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- It follows that the set $Z(\sigma) = \pi^{-1}(\sigma) = \{x; \Phi(x, \sigma) = 0\}$ is for any $\sigma \in \Sigma$ a smooth hypersurface in $X$,

- for any point $x \in X$ and for any tangent hyperplane $h \subset T_x(X)$ there is a locally unique hypersurface $Z(\sigma)$ through $x$ tangent to $h$. 
Funk-Radon transform

Let $X$ is an open set in an Euclidean $n$ space, $dV$ be the volume form.
Funk-Radon transform

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The quotient $dV/d\Phi$ denotes an arbitrary $n-1$ form $q$ such that $d\Phi \wedge q = dV$. 

Comparison with an Euclidean hypersurface integral:

$$M_{\Phi} f (\sigma) = \int \delta (\Phi (x, \sigma)) f dS_{jr} x \Phi (x, \sigma)$$

If $\mathbf{jr}_x \Phi (x, \sigma) = m(x) \mu (\sigma)$ for some continuous functions $m$ in $X$ and $\mu$ in $\Sigma$, then

$$\int \delta (\Phi (x, \sigma)) f dS = \mu (\sigma) M_{\Phi} (mf)(\sigma), \sigma \in \Sigma,$$
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Comparison with an Euclidean hypersurface integral:

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If $|\nabla_x \Phi(x, \sigma)| = m(x) \mu(\sigma)$ for some continuous functions $m$ in $X$ and $\mu$ in $\Sigma$, then

$$\int_{Z(\sigma)} f dS = \mu(\sigma) M_\Phi (mf)(\sigma), \sigma \in \Sigma$$
Further assumptions

- \( \Phi \) is called *resolved* if \( \Sigma = \mathbb{R} \times S^{n-1} \) and
  \[
  \Phi(x; p, \omega) = \theta(x, \omega) - p, \quad p \in \mathbb{R}, \quad \omega \in S^{n-1}
  \]
  for a smooth function \( \theta \) on \( X \times S^{n-1} \).
Further assumptions

- $\Phi$ is called resolved if $\Sigma = \mathbb{R} \times S^{n-1}$ and $\Phi(x; p, \omega) = \theta(x, \omega) - p$, $p \in \mathbb{R}$, $\omega \in S^{n-1}$ for a smooth function $\theta$ on $X \times S^{n-1}$.

- Acquisition geometry is called elliptic if (i) $\nabla_x \theta \wedge (d\omega \nabla_x \theta)^{n-1} \neq 0$ and
Further assumptions

- \( \Phi \) is called \textit{resolved} if \( \Sigma = \mathbb{R} \times S^{n-1} \) and 
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  for a smooth function \( \theta \) on \( X \times S^{n-1} \).

- Acquisition geometry is called \textit{elliptic} if (i) \( \nabla_x \theta \wedge (d_\omega \nabla_x \theta)^{n-1} \neq 0 \) and

- (ii) \textit{there are no conjugate points}, that is the equations 
  \( \theta(x, \omega) - \theta(y, \omega) = 0 \) and 
  \( d_\omega (\theta(x, \omega) - \theta(y, \omega)) = 0 \) are fulfilled 
  for no \( x \neq y \in X, \omega \in S^{n-1} \).
Let $f$ be a real smooth function in a manifold $X$, $n$ is natural. Define

$$I_n(\rho) = \int_X \frac{\rho}{(f+i0)^n} = \lim_{\varepsilon \searrow 0} \int_X \frac{\rho}{(f+i\varepsilon)^n},$$

for a smooth real density $\rho$ with compact support. If $df \neq 0$ on the zero set of $f$, then the limits exist and $I_n$ is a generalized function in $X$. 
Singular integrals

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- The functional
  \[ (P) \int_X \frac{\rho}{f^n} \equiv \text{Re} \, I_n(\rho) \]
  is called a principal value integral.
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  is called a *principal value* integral.

- For a resolved regular generating function $\Phi = \theta - p$ we define
  \[ \Theta_n(x, y) = (-1)^{n/2} \int_{S^{n-1}} \frac{\Omega}{(\theta(x, \omega) - \theta(y, \omega) - i0)^n}, \quad x \neq y \]
  where $\Omega$ is the Euclidean volume form on the unit sphere $S^{n-1}$. The singular integral converges by (ii).
**Theorem.** Let $\Phi = \theta - p$ be a resolved regular generating function, $f \in L_{2\text{comp}}(X)$ and $g = Mf$. If

$$\text{Re} \Theta_n(x, y) = 0$$

for $x \neq y \in X$, then for even $n$

$$f = \frac{1}{2D_n(x)} R^* \left( Hg^{(n-1)} \right)$$

and for odd $n$

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where

$$R^* g(x) = \int_{S^{n-1}} g(\theta(x, \omega), \omega) \Omega$$

is back projection, and
\[ g^{(n-1)} = \left( \frac{\partial}{2\pi i \partial p} \right)^{n-1} M_{\Phi} f, \quad H_a(p) = \frac{i}{\pi} \int \frac{a(q)}{p-q} dq \]

and

\[ D_n(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \frac{\Omega}{|\nabla_x \theta(x, \omega)|^n} \]
\[ g^{(n-1)} = \left( \frac{\partial}{2\pi i \partial p} \right)^{n-1} M_{\Phi} f, \quad Ha(p) = \frac{i}{\pi} \int \frac{a(q)}{p-q} dq \]

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- Inversion integral transforms converge in $L^2_{\text{loc}}$.
- Compare with the inversion formulas for Radon’s!
Checking the key condition

If \( n = 2 \) and \( \theta(x, \omega) - \theta(y, \omega) \) is for any \( x, y \in X \), \( x \neq y \) a trigonometric polynomial in \( \omega \) of positive degree with only real zeros then \( \text{Re} \Theta_2(x, y) = 0 \).
Checking the key condition

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- For any \( n \geq 3, v \in \mathbb{R}^n \) and \( a \in R \) be such that \( |a| < |v| \). Then

\[
\text{Re} \left( -1 \right)^{n/2} \int_{S^{n-1}} \frac{\Omega}{\left( \langle \omega, v \rangle - a - i0 \right)^n} = 0
\]
Examples

- **Radon transform.** For a generating function
  \[ \Phi(x; p, \omega) = \langle \omega, x \rangle - p \]
  defined in \( \mathbb{R}^n \times \Sigma \), \( \Sigma = \mathbb{R} \times S^{n-1} \) we have \( |\nabla \theta| = D_n(x) = 1 \).
Examples

- **Radon transform.** For a generating function \( \Phi(x; p, \omega) = \langle \omega, x \rangle - p \) defined in \( \mathbb{R}^n \times \Sigma \), \( \Sigma = \mathbb{R} \times S^{n-1} \) we have \( |\nabla \theta| = D_n(x) = 1 \).

- **Funk transform.** If \( n = 2 \) is even then any function \( f \) can be reconstructed from its integrals \( g(\sigma) \) over big circles \( S(\sigma) = \{ x \in X, \langle \sigma, x \rangle = 0 \} \), \( \sigma \in S^2_+ \) by

\[
  f(x) = \frac{1}{4\pi^2} \int_{S^+} \frac{g(\sigma) \, d\sigma}{\langle \sigma, x \rangle^2}
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where \( S^{n-1}_+ = \{ \sigma \in \mathbb{R}^{n+1}; |\sigma| = 1, \sigma_0 \geq 0 \} \) is a hemisphere.
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- For \( n = 3 \) we have

  \[
  f(x) = -\frac{1}{8\pi^2} \int_{S^{n-1}_+} \delta''(\langle \sigma, x \rangle) \, g(\sigma) \, d\sigma
  \]
**Fully geodesic surfaces.** Take the generating function
\[ \theta = -2 \langle \omega, x \rangle \left( |x|^2 + 1 \right)^{-1} \]
in the unit ball \( X \subset \mathbb{R}^n \) and set
\[ g(\sigma) = \int_{Z(\sigma)} f(x) \, d_g S \]
where hypersurfaces \( Z(\sigma) = Z(p, \omega) \), \(-1 < p < 1\) are fully geodesics for the hyperbolic metric \( d_g s = 2 \left( 1 - |x|^2 \right)^{-1} ds \).
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**Reconstruction for** \( n = 2 \) is
\[ f(x) = \frac{1}{4\pi^2} \int_{Q_+} g(\sigma) \, dg \sigma \]
and for \( n = 3 \)
\[ f(x) = -\frac{1}{8\pi^2} \int_{Q_+} \delta'' \left( \langle \sigma, x \rangle \right) g(\sigma) \, dg \sigma \]
where \( Q_+ = \left\{ \sigma = (\sigma_0, \sigma') \in \mathbb{R}^{n+1}; \sigma_0^2 - |\sigma'|^2 = -1, \sigma_0 \geq 0 \right\} \) is the dual one sheet hyperboloid.
Equidistant spheres. Let $X$ be again a unit $n$ ball, $n \geq 2$ and $\theta (x, \omega) = (r - \langle \omega, x \rangle) \left(1 - |x|^2\right)^{-1}, \omega \in S^{n-1}, 0 \leq r < 1.$
- **Equidistant spheres.** Let $X$ be again a unit $n$ ball, $n \geq 2$ and $\theta(x, \omega) = (r - \langle \omega, x \rangle) \left(1 - |x|^2\right)^{-1}$, $\omega \in S^{n-1}$, $0 \leq r < 1$.

- **Horospheres.** Take $r = 1$ in the above formula.
• **Equidistant spheres.** Let $X$ be again a unit $n$ ball, $n \geq 2$ and $	heta(x, \omega) = (r - \langle \omega, x \rangle) \left(1 - |x|^2\right)^{-1}$, $\omega \in S^{n-1}$, $0 \leq r < 1$.

• **Horospheres.** Take $r = 1$ in the above formula.

• **Spheres** centered on the boundary, All 4 type of sphere families
Let $\xi : S^{n-1} \to \mathbb{R}^n$ be a smooth map and generating function and $\theta (x, \omega) = |x - \xi (\omega)|^2$, $\omega \in S^{n-1}$ be a generating function.
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The equation $\Phi = 0$ defines the family of spheres with the centers $\xi (\omega)$, $\omega \in S^{n-1}$. The image $C$ of $\xi$ is called central set.
Let $\zeta : S^{n-1} \to \mathbb{R}^n$ be a smooth map and generating function and $\theta (x, \omega) = |x - \zeta (\omega)|^2$, $\omega \in S^{n-1}$ be a generating function. The equation $\Phi = 0$ defines the family of spheres with the centers $\zeta (\omega)$, $\omega \in S^{n-1}$. The image $C$ of $\zeta$ is called central set. This geometry is of special interest in view of application to the photoacoustic (thermoacoustic) tomography.
Let $\xi: S^{n-1} \rightarrow \mathbb{R}^n$ be a smooth map and generating function and $\theta(x, \omega) = |x - \xi(\omega)|^2$, $\omega \in S^{n-1}$ be a generating function. The equation $\Phi = 0$ defines the family of spheres with the centers $\xi(\omega)$, $\omega \in S^{n-1}$. The image $C$ of $\xi$ is called central set.

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Photoacoustic geometries (cont.)

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- **Paraboloids**
Hyperbolic curves

- In the case $n = 2$ there are more geometries which allow exact reconstruction formulas.
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A curve $\mathbf{C} \subset \mathbb{R}^2$ is called *trigonometric* of degree $k$ if it is given by a parametric equation

$$x_1 = \xi_1(\varphi), x_2 = \xi_2(\varphi), \varphi \in S^1$$

where $\xi_1, \xi_2$ are real trigonometric polynomials of degree $k$. 
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The set $H$ of all hyperbolic points is always closed and convex. We call a curve $\mathbf{C}$ *hyperbolic* if the set $H$ of hyperbolic points is not empty.
Proposition. For arbitrary points \( x, y \in H, \ x \neq y \), all roots of the polynomial \( |x - \xi(\omega)|^2 - |y - \xi(\omega)|^2 \) (of order \( k \)) are real.
Hyperbolic central sets

- **Proposition.** For arbitrary points $x, y \in H$, $x \neq y$, all roots of the polynomial $|x - \xi(\omega)|^2 - |y - \xi(\omega)|^2$ (of order $k$) are real.

- **Corollary.** If $C$ is a hyperbolic trigonometric curve, then the FBP reconstruction holds arbitrary function $f$ supported in the set $H$ of hyperbolic points from data of integrals

$$g(\sigma) = \int_{\sigma} f \, ds$$

over circles $\sigma$ centered at $C$. 
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Remind Retrowsky's theory of lacunas of fundamental solutions of hyperbolic equations!
1. \( \xi_1(\varphi) = 2 \cos 2\varphi - \cos \varphi, \quad \xi_2(\varphi) = 2 \sin 2\varphi + \sin \varphi \)

The hyperbolic set \( H \) is the triangle in the middle, \( k = 2 \).
2. A hyperbolic "square" set is defined by the trigonometric curve
\[ \xi_1(\varphi) = 2\cos 3\varphi + \cos \varphi, \quad \xi_2(\varphi) = 2\sin 3\varphi - \sin \varphi \]
3. A "pentagon" is the hyperbolic set of the curve
\[ \tilde{\xi}_1(\varphi) = 5 \cos 4\varphi + 4 \cos \varphi, \quad \tilde{\xi}_2(\varphi) = 5 \sin 4\varphi - 4 \sin \varphi, \quad k = 4 : \]
Thank you for your attention!
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