Detection of unknown boundaries and inclusions in elastic plates

A. Morassi[1] E. Rosset[2] S. Vessella[3]

[1]Dipartimento di Ingegneria Civile e Architettura,
Università di Udine.

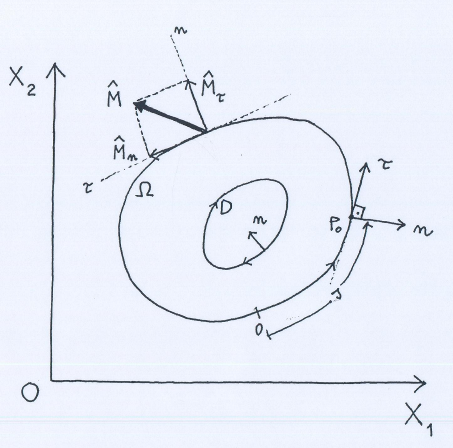
[2]Dipartimento di Matematica e Geoscienze
Università di Trieste

[3]DIMAD
Università di Firenze

Linkoping, IPA 2013

KINDS OF DEFECTS:

- Inclusions made of elastic material either softer or harder than the background
- Cavities
- Inaccessible portions of the boundary of the plate
- Rigid inclusions



$$\hat{M} = \hat{M}_\tau n + \hat{M}_n \tau = \hat{M}_2 e_1 + \hat{M}_1 e_2, \quad \text{on } \partial\Omega, \qquad \tau = e_3 \times n$$

Statical equilibrium problem when D is a rigid inclusion

$$(BVP)\begin{cases} div(div(\mathbb{P}\nabla^2 w)) = 0, & \text{in } \Omega\setminus\overline{D},\\ (\mathbb{P}\nabla^2 w)n\cdot n = -\hat{M}_n, & \text{on } \partial\Omega,\\ div(\mathbb{P}\nabla^2 w)\cdot n + ((\mathbb{P}\nabla^2 w)n\cdot\tau)_{,s} = (\hat{M}_\tau)_{,s}\,, & \text{on } \partial\Omega,\\ w|_{\overline{D}}\in\mathcal{A}, & \\ w^e{}_{,n} = w^i{}_{,n}\,, & \text{on } \partial D,\end{cases}$$

coupled with the equilibrium conditions

$$(E)\int_{\partial D}\left(\mathrm{div}(\mathbb{P}\nabla^2 w)\cdot n + ((\mathbb{P}\nabla^2 w)n\cdot\tau)_{,s}\right)g - ((\mathbb{P}\nabla^2 w)n\cdot n)g_{,n} = 0,$$

$$\forall g\in\mathcal{A},$$

$$\mathcal{A} = \{g(x_1,x_2) = ax_1 + bx_2 + c,\ a,b,c\in\mathbb{R}\}$$

- $\mathbb{P} = \frac{h^3}{12}\mathbb{C}$
- $\mathbb{C} \in L^\infty(\overline{\Omega})$, $C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta} = C_{\gamma\delta\beta\alpha}$, $\quad \alpha, \beta, \gamma, \delta = 1, 2$.
- $\mathbb{C}$ **strongly convex** :

$$\exists\, \gamma > 0 \mid \mathbb{C}(x)A \cdot A \geq \gamma |A|^2,$$

  for every 2×2 symmetric matrix A and for every $x \in \overline{\Omega}$.
- $\hat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$.
- Compatibility conditions : $\int_{\partial\Omega} \hat{M}_\alpha = 0$, $\alpha = 1, 2$,
- Ω, D simply connected bounded domains in $\mathbb{R}^2$ of class $C^{1,1}$, $D \subset\subset \Omega$.

DIRECT PROBLEM

Problem (BVP) admits a weak solution $w \in H^2(\Omega \setminus \overline{D})$, which is determined up to the addition of an affine function. If we require

$$(NC) \qquad w = 0, \quad w_{,n} = 0 \qquad \text{on } \partial D.$$

there exists a unique solution satisfying

$$\|w\|_{H^2(\Omega\setminus\overline{D})} \leq C\|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}.$$

Let $\quad H = \{v \in H^2(\Omega \setminus \overline{D}) \mid v = 0, v_{,n} = 0 \text{ on } \partial D\}$

Variational formulation A weak solution to the mixed problem $(BVP) - (NC)$ is a function $w \in H$ satisfying

$$\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla^2 w \cdot \nabla^2 v = \int_{\partial\Omega} -\hat{M}_{\tau,s} v - \hat{M}_n v_{,n}\,, \qquad \forall v \in H.$$

INVERSE PROBLEM

To determine the **unknown** rigid inclusion D from the additional measurement taken on an open portion Γ of $\partial\Omega$ of the Dirichlet data $\{w, w_{,n}\}$, that is from the Cauchy data on Γ :

$$(Cauchy)\begin{cases} w|_{\Gamma} \\ w_{,n}|_{\Gamma} \\ (\mathbb{P}\nabla^2 w)n \cdot n|_{\Gamma} = -\widehat{M}_n \\ div(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s}|_{\Gamma} = M_{\tau,s} \end{cases}$$

APPLICATIONS

Non-destructive testing for quality assessment of materials

HYPOTHESES FOR UNIQUENESS

- $(\widehat{M}_n, (\widehat{M}_\tau)_{,s}) \not\equiv 0$
- The rigid inclusions D is of class $C^{3,1}$.
- The boundary measurements are taken on an open portion Γ of $\partial\Omega$ of class $C^{3,1}$.
- The elasticity tensor $\mathbb{C}$ is of class $C^{1,1}$ and satisfies a **dichotomy condition** ensuring that the complex characteristic lines of the principal part of the operator have the same multiplicity everywhere (1 or 2), that is:

$$(1) \qquad \text{either} \quad \mathcal{D}(x) > 0, \ \forall x \in \mathbb{R}^2$$

$$(2) \qquad \text{or} \quad \mathcal{D}(x) = 0, \ \forall x \in \mathbb{R}^2$$

$\mathcal{D}(x)$ absolute value of the discriminant of $p(x;(t,1))$, where $p(x;\xi)$ is the symbol of the principal part of the operator.

Class of Orthotropic materials: through each point there pass three mutually orthogonal planes of elastic symmetry, which are parallel at all points.

- Any orthotropic material satisfy either (1) or (2)
- Isotropic material $\Rightarrow \mathcal{D}(x) \equiv 0$
- There exist anisotropic orthotropic materials for which $\mathcal{D}(x) \equiv 0$

Remark. The value of $\mathcal{D}(x)$ cannot be interpreted as a measure of anisotropy.

$$(1) \qquad \text{either} \quad \mathcal{D}(x) > 0,\ \forall x \in \mathbb{R}^2$$

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Theorem (A. Morassi, E.R., C.R. Mecanique 338, 2010)

Let w_i, $i = 1, 2$, be the solutions to the normalized problem

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(NP)\begin{cases}
div(div(\mathbb{P}\nabla^2 w_i)) = 0, & \text{in } \Omega \setminus \overline{D_i},\\
(\mathbb{P}\nabla^2 w_i)n \cdot n = -\hat{M}_n, & \text{on } \partial\Omega,\\
div(\mathbb{P}\nabla^2 w_i) \cdot n + ((\mathbb{P}\nabla^2 w_i)n \cdot \tau)_{,s} = (\hat{M}_\tau)_{,s}\,, & \text{on } \partial\Omega,\\
w_i = 0 & \text{on } \partial D_i,\\
w_{i,n} = 0 & \text{on } \partial D_i.
\end{cases}
$$

If for some $g \in \mathcal{A}$

$$
w_1 - w_2 = g, \qquad (w_1 - w_2)_{,n} = g_{,n}\,, \qquad on\ \Gamma,
$$

then $D_1 = D_2$.

OTHER KINDS OF DEFECTS

- **CAVITIES** $\rightarrow$ uniqueness with TWO measurements (Morassi, E. R., Vessella, Inverse Problems and Imaging, 2007)
- **UNKNOWN BOUNDARIES** $\rightarrow$ uniqueness with one measurement (A. Morassi, E. R., C.R. Mecanique, 2010)

Math. In both cases homogeneous Neumann b.c. on the boundary of the defect, but different geometry.

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A PRIORI ASSUMPTIONS FOR STABILITY

- $\partial\Omega$ of class $C^{2,1}$ with constants ρ_0 , M_0
- $|\Omega| \leq M_1$
- $\text{dist}(D, \partial\Omega) \geq \rho_0$
- ∂D of class $C^{3,1}$ with constants ρ_0 , M_0
- $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$, $(\widehat{M}_n, (\widehat{M}_\tau)_{,s}) \not\equiv 0$
- $\text{supp}(\widehat{M}) \subset \Gamma$, $\Gamma \in C^{3,1}$
- $\partial\Omega \cap B_{\rho_0}(P_0) \subset \Gamma$, for some $P_0 \in \Gamma$
- $\dfrac{\|\widehat{M}\|_{L^2}}{\|\widehat{M}\|_{H^{-1/2}}} \leq F$

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- $\|\mathbb{C}\|_{C^{1,1}}(\mathbb{R}^2) \leq M$
- $\mathbb{C}A \cdot A \geq \gamma |A|^2$, $\forall$ symmetric 2×2 matrix A
- Dichotomy condition:

$$\text{either} \quad \mathcal{D}(x) > 0,\ \forall x \in \mathbb{R}^2\ (\delta_1 = \min_{\mathbb{R}^2} \mathcal{D})$$

$$\text{or} \quad \mathcal{D}(x) = 0,\ \forall x \in \mathbb{R}^2$$

where $\mathcal{D}(x)$ is the discriminant of the polynomial $p(x;(t,1))$, where $p(x;\xi)$ is the symbol of the principal part of the operator.

Theorem (Stability, A. Morassi, E. R., S. Vessella, SIAM J. Math. Anal. 2012)

Let $w_i \in H^2(\Omega \setminus \overline{D_i})$ be the solutions to (NP), $i = 1, 2$.
If, for some $g \in \mathcal{A}$,

$$\|w_1 - w_2 - g\|_{L^2(\Gamma)} + \|(w_1 - w_2),_n - g,_n\|_{L^2(\Gamma)} \leq \epsilon,$$

then

$$d_{\mathcal{H}}(\overline{D_1}, \overline{D_2}) \leq C(\log|\log \epsilon|)^{-\eta},$$

for every ϵ, $0 < \epsilon < e^{-1}$, where $C > 0$, η, $0 < \eta \leq 1$, are constants only depending on the a priori data.

Stability results in other contexts

- **Electrical conductors: log** - type stability estimate for perfectly insulating and for perfectly conducting inclusions
  $n = 2$:
  Bukhgeim, Cheng, Yamamoto 1998, 1999, 2000,
  Beretta, Vessella 1998,
  Rondi 1999,
  Alessandrini, Rondi 2001;
  $n \geq 2$:
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- **Thermic conductors: log** - type stability estimate for solidification fronts and cavities
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!!! The log rate of convergence is **optimal**

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Remark The weaker rate of convergence in elastostatics is due to the lack of quantitative estimates of unique continuation at the boundary for solutions satisfying homogeneous Neumann/Dirichlet boundary conditions.

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MATHEMATICAL TOOLS FOR UNIQUENESS

- **Regularity of solutions**
  For every solution w to the normalized problem (NP),
  $w \in C^1$ up to ∂D
  the b.c. at ∂D hold in the classical sense.
- **Three spheres inequality**

$$\Downarrow$$

- **Weak Unique Continuation Property of the solution.** If $w \equiv 0$ in some open nonempty subset of $\Omega \setminus \overline{D}$, then $w \equiv 0$ in $\Omega \setminus \overline{D}$.
- **Uniqueness for the Cauchy problem**

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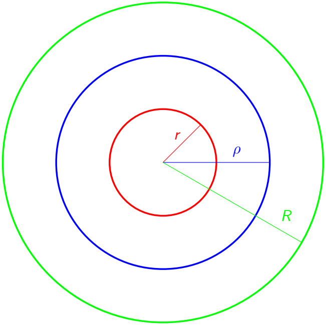
THREE SPHERES INEQUALITY

$$\int_{B_\rho} |\nabla^2 w|^2 \leq C \left(\int_{B_r} |\nabla^2 w|^2 \right)^{\theta} \left(\int_{B_R} |\nabla^2 w|^2 \right)^{1-\theta}$$



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Lipschitz propagation of smallness

Let $w \in H^2(\Omega \setminus \overline{D})$ be a solution to the normalized problem (NP). There exists $s > 1$ (only depending on γ_0, M, δ_1, M_0) s.t. $\forall \rho > 0$ and $\forall \overline{x} \in (\Omega \setminus \overline{D})_{s\rho}$ we have

$$(LPS) \qquad \int_{B_\rho(\overline{x})} |\nabla^2 w|^2 \geq \frac{C}{\exp\left[A\rho^{-B}\right]} \int_{(\Omega \setminus \overline{D})} |\nabla^2 w|^2,$$

where $A > 0$, $B > 0$ and $C > 0$ only depend on the a-priori data.

"INTERMEDIATE CASE": ELASTIC INCLUSIONS

The inclusion D is made by elastic material with strongly convex tensor $\widetilde{\mathbb{C}}$

- Uniqueness results not available

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- Alternative approach: estimates of the area of the inclusion

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JUMP CONDITION

$$\exists\, \delta > 0 \text{ s.t. } \textit{either } (\widetilde{\mathbb{C}} - \mathbb{C}) \geq \delta\mathbb{C}, \quad \textit{or } (\mathbb{C} - \widetilde{\mathbb{C}}) \geq \delta\mathbb{C}.$$

FATNESS CONDITION

D measurable such that here exists $h_1 > 0$ s.t.

$$area(D_{h_1}) \geq \frac{1}{2} area(D),$$

where $D_{h_1} = \{x \in D \mid dist(x, \partial D) > h_1\}$.

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Let $w \in H^2(\Omega)$ be the solution to the equilibrium problem when the inclusion D is present

$$
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div(div((\chi_{\Omega\setminus D}\mathbb{P} + \chi_D\widetilde{\mathbb{P}})\nabla^2 w)) = 0, & \text{in } \Omega,\\
(\mathbb{P}\nabla^2 w)n\cdot n = -\hat{M}_n, & \text{on } \partial\Omega,\\
div(\mathbb{P}\nabla^2 w)\cdot n + ((\mathbb{P}\nabla^2 w)n\cdot\tau)_{,s} = (\hat{M}_\tau)_{,s}\,, & \text{on } \partial\Omega,
\end{cases}
$$

normalized by the conditions

$$
\int_\Omega w = 0, \quad \int_\Omega \nabla w = 0.
$$

Let $w_0 \in H^2(\Omega)$ be the solution to the equilibrium problem when the inclusion is absent

$$(P_0) \begin{cases} div(div(\mathbb{P}\nabla^2 w_0)) = 0, & \text{in } \Omega, \\ (\mathbb{P}\nabla^2 w_0)n \cdot n = -\hat{M}_n, & \text{on } \partial\Omega, \\ div(\mathbb{P}\nabla^2 w_0) \cdot n + ((\mathbb{P}\nabla^2 w_0)n \cdot \tau),_s = (\hat{M}_\tau),_s \,, & \text{on } \partial\Omega, \end{cases}$$

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Works exerted by $\widehat{M}$ when D is present and absent, respectively:

$$W = \int_\Omega (\mathbb{P} + (\widetilde{\mathbb{P}} - \mathbb{P})\chi_D)\nabla^2 w \cdot \nabla^2 w = -\int_{\partial\Omega} \widehat{M}_{\tau,s} w + \widehat{M}_n w_{,n}$$

$$W_0 = \int_\Omega \mathbb{P}\nabla^2 w_0 \cdot \nabla^2 w_0 = -\int_{\partial\Omega} \widehat{M}_{\tau,s} w_0 + \widehat{M}_n w_{0,n}$$

$$\frac{|W - W_0|}{W_0}$$ **normalized work gap**

Theorem (Size estimates, A. Morassi, E.R., S. Vessella, DCDS S 6, 2013)

Let $\mathbb{C}$ satisfy the dichotomy condition.

$$C_1 \frac{|W - W_0|}{W_0} \leq area(D) \leq C_2 \frac{|W - W_0|}{W_0}$$

Remark: The fatness condition is not needed for the lower bound. If the background material is isotropic (or $\mathcal{D}(x) \equiv 0$) a (weaker) upper estimate holds without fatness condition

$$area(D) \leq C_2 \left(\frac{|W - W_0|}{W_0} \right)^{\frac{1}{p}},$$

where $p > 1$. (A. Morassi, E.R., S. Vessella, Inverse Problems 25, 2009)

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Size estimates for estreme inclusions

(cavities and rigid inclusions)

Upper Bounds: same results
Lower bounds: under additional assumptions on the regularity of the defect, a weaker estimate is derived

$$area(D) \geq C_2 \left(\frac{|W - W_0|}{W_0} \right)^2 ,$$

(A. Morassi, E.R., S. Vessella, J. Inverse Ill Posed Problems 2013, review paper)

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