Ergodicity of branching billiards

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Quantum ergodicity

- M is a compact n-dimensional domain (or Riemannian manifold) with smooth boundary ∂M.
- ϕ_j are the orthonormalized eigenfunctions of the Laplace operator Δ on M with D or N boundary condition.

One says that quantum ergodicity holds if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left| \langle Q\phi_j, \phi_j \rangle - \int_{S^*M} \sigma_Q(x,\xi)\,\mathrm{d}\omega \right|^2 = 0 \qquad (*)$$

for every pseudodifferential operator Q, where σ_Q is the principal symbol of Q and $\mathrm{d}\omega = \frac{\mathrm{d}x\,\mathrm{d}\xi}{\mathrm{Vol}\,(S^*M)}$ is the normalized measure on the phase space $S^*M = \{x \in M, \xi \in \mathbb{S}^{n-1}\}$.

The quantum ergodicity theorem states that (*) is true whenever the billiard flow in the phase space is ergodic. Various versions of this theorem were proved in papers by Shnirelman, Colin de Verdière, Helffer–Martinez–Robert, Zelditch-Zworski and others.

Our model

- $\Sigma \subset M$ be a closed smooth surface of codimension one, which splits M into two disjoint parts M_+ and M_-.
- A is the differential operator such that $A|_{M\pm} = c_\pm \Delta$ where c_+ and c_- are nonnegative constants.

The domain of A consists of functions u from the Sobolev space $H^2(M \setminus \Sigma)$, satisfying the D or N boundary condition on ∂M and the transmission 'boundary' conditions on Σ

$$u_+|_\Sigma = u_-|_\Sigma \quad \text{and} \quad c_+ \partial_n u_+|_\Sigma = -\, c_- \partial_n u_-|_\Sigma \,,$$

where $u_\pm = u|_{M\pm}$ and ∂_n is the normal derivative.

Our main goal was to find out under what assumptions on the associated classical dynamics we have quantum ergodicity.

Remark. Branching billiards also occur in problems described by systems of partial differential equations, higher order equations, and on quantum graphs. Our results can be carried over in a straightforward manner to those situations.

Propagation of singularities

It is known that singularities of solutions to the wave equation $\partial_t^2 u(t,x) - Au(t,x)$ propagates along billiard trajectories. The trajectory emanating from a fixed point in a fixed direction is uniquely defined until it hits the set Σ. Then, generally speaking, it splits into two geodesics according to the standard law of geometric optics. One of them is obtained by reflection, the other is the refracted trajectory which goes through Σ but changes its direction. For some angles of incidence the refracted trajectory does not exists, and then one says that there is full inner reflection.

The trajectory obtained by consecutive reflections and/or refractions is called a billiard trajectory. Generally speaking, there are infinitely many billiard trajectories originating from a given point in a given direction. Moreover, the set of these trajectories is typically uncountable.

Classical dynamics of branching billiards: geometry

- $x^t_\varkappa(y,\eta)$ is the billiard trajectory originating from $y \in M$ in the direction η, where $\varkappa$ is an index specifying the type of trajectory.
- $(x^t_\varkappa(y,\eta), \xi^t_\varkappa(y,\eta))$ is the corresponding billiard trajectory in the phase space, where $\xi^t_\varkappa = \dot{x}^t_\varkappa$ is the (co)tangent vector.
- $\Phi^t_\varkappa$ is the shift in the phase space along the billiard trajectories $(x^t_\varkappa, \xi^t_\varkappa)$ of type $\varkappa$.

The mapping $\Phi^t_\varkappa : (y,\eta) \mapsto (x^t_\varkappa(y,\eta), \xi^t_\varkappa(y,\eta))$ is a homogeneous canonical transformation in the sense of symplectic geometry $\Longrightarrow$ it preserves the measure $\mathrm{d}\omega$ on S^*M.

Definition. The branching billiard system as a family of multi-valued canonical transformations Φ^t, mapping $(y,\eta) \in S^*M$ into the set $\Phi^t(y,\eta) = \bigcup_\varkappa \Phi^t_\varkappa(y,\eta)$.

Bad trajectories

A billiard trajectory is not well-defined if

- it hits $\Sigma \cup \partial M$ infinitely many times in a finite time, or
- the angle of incidence or the angle of refraction is zero.

Trajectories of the first type are called *dead-end*, trajectories of the second type are said to be *grazing*.

The set of starting points $(y,\eta) \in S^*M$ of grazing trajectories has measure zero. However, there are reasons to believe that the set of starting points of dead-end trajectories may have a positive measure.

- $\mathcal{O}_T$ is the set of points $(y,\eta) \in S^*M$ such that all the billiard trajectories $(x_\varkappa^t(y,\eta), \xi_\varkappa^t(y,\eta))$ of length T are well defined and experience only finitely many reflections and refractions.

Assumption. $\mathcal{O}_T$ is a set of full measure in S^*M for each $T > 0$.

$\Longrightarrow$ the multi-valued canonical transformations Φ^t are defined on a set of full measure in S^*M for all $t \geqslant 0$.

The unitary group $U(t) = e^{-itA^{1/2}}$

Let Q be a pseudodifferential operator with supp $Q \subset \mathcal{O}_T$. Then, on the time interval $[0, T)$, we have $U(t)Q = \sum_\varkappa U_\varkappa(t)Q$ modulo an integral operator with an infinitely smooth kernel, where $U_\varkappa(t)$ are Fourier integral operators (FIOs) associated with $\Phi^t_\varkappa$. The principal symbol $\sigma_\varkappa(t; y, \eta)$ of the FIO $U_\varkappa(t)$ is the function on the billiard trajectory $(x^t_\varkappa(y, \eta), \xi^t_\varkappa(y, \eta))$ defined as follows.

(1) $\sigma_\varkappa$ is constant on every segment of $(x^t_\varkappa, \xi^t_\varkappa)$ and $\sigma_\varkappa \equiv 1$ on the first segment.

(2) If $(x^t_\varkappa, \xi^t_\varkappa)$ hits Σ at a point (x, ξ) where there exist reflected and refracted trajectories, then $\sigma_\varkappa$ is multiplied either by a reflection coefficient $\tau'(x, \xi)$ or by a refraction coefficient $\tau''(x, \xi)$, where τ' and τ'' are real numbers such that $(\tau')^2 + (\tau'')^2 = 1$.

(3) If $(x^t_\varkappa, \xi^t_\varkappa)$ hits the boundary ∂M or the surface Σ at a point of full inner reflection (x, ξ) then $\sigma_\varkappa$ is multiplied by a complex number $\tau(x, \xi)$ such that $|\tau(x, \xi)| = 1$.

The coefficients $\tau'(x, \xi)$, $\tau''(x, \xi)$ and $\tau(x, \xi)$ depend on the angle of incident and can be explicitly evaluated.

Propagation of energy

We call the number $|\sigma_\varkappa(t; y, \eta)|^2$ the weight of the trajectory $(x^s_\varkappa(y,\eta), \xi^s_\varkappa(y,\eta))$, $s \in [0,t]$. It can be thought of as the proportion of energy transmitted along the billiard trajectory, or the probability for a particle to travel along this trajectory.

Since $(\tau'(x,\xi))^2 + (\tau''(x,\xi))^2 = 1$ and $|\tau(x,\xi)| = 1$, we have the following conservation of energy law:

$$\sum_{(x^t_\varkappa, \xi^t_\varkappa)} |\sigma_\varkappa(t, y, \eta)|^2 \;=\; 1\,,$$

where the sum is taken over all distinct billiard trajectories of 'length' t originating from (y,η).

The geometric definition of the branching billiard system does not take into account the propagation of energy and, therefore, is not sufficient for the study of 'quantum' properties of A.

Classical dynamics of branching billiards: weights

- $w^{\mathrm{c}}_{(x,\xi)}(t,y,\eta) = \sum_{(x^t_\varkappa,\xi^t_\varkappa)} |\sigma_\varkappa(t,y,\eta)|^2$,
- $w^{\mathrm{d}}_{(x,\xi)}(t,y,\eta) = \left|\sum_{(x^t_\varkappa,\xi^t_\varkappa)} \sigma_\varkappa(t,y,\eta)\,\right|^2$,

where the sums are taken over all distinct billiard trajectories originating from (y,η) and ending at (x,ξ) at the time t.

- $w^{\mathrm{c}}_{(x,\xi)}$ can be thought of as the proportion of energy transmitted from (y,η) into (x,ξ) along all trajectories.
- $w^{\mathrm{d}}_{(x,\xi)}$ does not seem to admit a simple physical interpretation.

The difference between $w^{\mathrm{d}}_{(x,\xi)}$ and $w^{\mathrm{c}}_{(x,\xi)}$ is in the contributions from recombining billiard trajectories, i.e. such trajectories that $(x^t_\kappa,\xi^t_\kappa) = (x^t_{\kappa'},\xi^t_{\kappa'})$ but $(x^s_\kappa,\xi^s_\kappa) \neq (x^s_{\kappa'},\xi^s_{\kappa'})$ for some $s \in (0,t)$.

$$\sum_{(x,\xi)\in\Phi^t(y,\eta)} w^{\mathrm{d}}_{(x,\xi)}(t,y,\eta) \;=\; \sum_{(x,\xi)\in\Phi^t(y,\eta)} w^{\mathrm{c}}_{(x,\xi)}(t,y,\eta) \;=\; 1$$

The latter inequality is obvious. The former is an inverse result.

Classical dynamics of branching billiards: operators

Definition. The classical transfer operators Θ_t^{c} and the diagonal transfer operators Θ_t^{d} in the space of L^∞-functions on S^*M are defined for times $t \geqslant 0$ by the equalities

$$(\Theta_t^{\mathrm{c}} f)(y,\eta) := \sum_{(x,\xi)\in\Phi^t(y,\eta)} w^{\mathrm{c}}_{(x,\xi)}(t,y,\eta)\, f(x,\xi)\,,$$

$$(\Theta_t^{\mathrm{d}} f)(y,\eta) := \sum_{(x,\xi)\in\Phi^t(y,\eta)} w^{\mathrm{d}}_{(x,\xi)}(t,y,\eta)\, f(x,\xi)\,.$$

Remark. Θ_t^{d} and Θ_t^{c} are uniformly bounded in the spaces $L^p(S^*M)$ and are isometries in $L^1(S^*M)$.

Remark. The operators Θ_t^{c} form a semigroup, whereas $\Theta_t^{\mathrm{d}}\Theta_s^{\mathrm{d}}$ may not coincide with $\Theta_{t+s}^{\mathrm{d}}$.

Ergodicity

Definition. Θ_t is ergodic if for all $f \in L^\infty(S^*M)$

$$2T^{-2}\int_0^T\int_0^t (\Theta_s f)(y,\eta)\,\mathrm{d}s\,\mathrm{d}t \;\to\; \int_{S^*M} f(y,\eta)\,\mathrm{d}\omega(y,\eta) \qquad (1)$$

as $T \to +\infty$ almost everywhere in S^*M.

Remark. The traditional definition of ergodicity assumes that

$$t^{-1}\int_0^t (\Theta_s f)\,\mathrm{d}s \;\to\; \int_{S^*M} f(y,\eta)\,\mathrm{d}\omega(y,\eta) \qquad (2)$$

as $t \to +\infty$ almost everywhere.

(2) $\Longrightarrow$(1) but, generally speaking, the converse is not true.
(2)$\Longleftrightarrow$(1) if there are no branching trajectories.

Main Theorem. Ergodicity of the diagonal dynamics Θ_s^{d} implies quantum ergodicity.

Sketch of proof

(a) Develop a symbolic calculus for FIOs. We need precise formulae for the symbols of compositions and adjoints, so that the usual statements like
"the composition is a FIO associated with the composition of the canonical transformations, whose principal symbol is a half-density with values in the Keller–Maslov bundle which can be calculated as explained on the pages ... of [...]*"*
do not help.

(b) Obtain an asymptotic formula for $\sum_{\lambda_j<\lambda}\langle V\phi_j,\phi_j\rangle$, where λ_j and ϕ_j are eigenvalues and eigenfunctions, and V is a FIO.

(c) Note that $\langle A\phi_j,\phi_j\rangle \;=\; \langle A_T\phi_j,\phi_j\rangle$, where $A_T := T^{-1}\,\int_0^T U(-t)AU(t)\mathrm{d}t$. Assume that the average of the principal symbol over S^*M is zero. Then it sufficient to show that $\limsup_{N\to\infty} N^{-1}\sum_{j=1}^N\langle A_T^*A_T\phi_j,\phi_j\rangle = 0$.

(d) Substitute the FIO representation for $U(t)$ and obtain an asymptotic formula for $\langle A_T^*A_T\phi_j,\phi_j\rangle$, using (a) and (b).