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Inverse Problems and Applications

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Analysis of a forward problem in optical tomography

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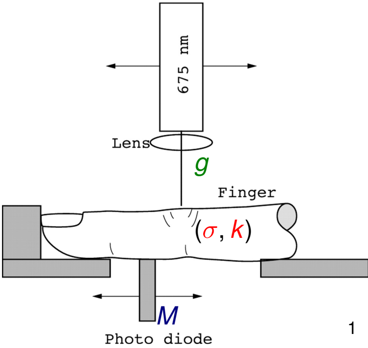
Optical tomography: Model & basic equations (Forward problem)

Radiative Transfer Equation (RTE)

$$v \cdot \nabla u + \sigma u = \int_V k(\cdot, \cdot, v')u(\cdot, v')\,\mathrm{d}v' + f$$

$$u(x, v) = g(x, v) \quad \text{if } n(x) \cdot v < 0$$

- σ attenuation coefficient
- k scattering kernel
- u photon density
- Observation $B : u \mapsto M := M(u)$ (linear)
- Forward operator $F : (\sigma, k) \mapsto M$ (nonlinear)



[1]

[1] Scheel et al, First clinical evaluation of sagittal laser OT for detection of synovitis in arthritic finger joints, *Ann Rheum Dis*, 2005;64:239-245

Solvability of the radiative transfer equation in L^p spaces

- Setting
- Solvability in L^∞
- Solvability in L^1
- Solvability in L^p
- Inverse Problem

Radiative Transfer Equation and basic assumptions

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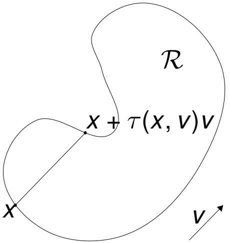
$$v \cdot \nabla u(x,v) + \sigma(x,v)u(x,v) = \int_V k(x,v,v')u(x,v')\,\mathrm{d}v' + f(x,v) \quad \text{in } \mathcal{R} \times V$$

$$u = 0 \quad \text{on } \Gamma_- := \{(x,v) \in \partial\mathcal{R} \times V : n(x) \cdot v < 0\}$$

- $\mathcal{R} \subset \mathbb{R}^d$ bounded and convex, $\partial\mathcal{R} \in C^{0,1}$, $V \subset \mathbb{R}^d$.
- $0 \leq \sigma\tau \in L^\infty(\mathcal{R} \times V)$, $k(x,v,v') \geq 0$

$$\sigma_p(x,v') := \int_V k(x,v,v')\,\mathrm{d}v \leq \sigma(x,v')$$

$$\sigma_p'(x,v) := \int_V k(x,v,v')\,\mathrm{d}v' \leq \sigma(x,v)$$



- weight $\omega := \max\{\sigma, \tau^{-1}\} > 0$ a.e.
  $L^p(\omega) = \{u : \omega|u|^p \in L^1(\mathcal{R} \times V)\}$, $L^\infty(\omega) = L^\infty$.
- $W^p(\omega) := \{u \in L^p(\omega) : \frac{1}{\omega} v \cdot \nabla u \in L^p(\omega)\}$

Solvability without scattering

$$v\nabla u + \sigma u = f, \qquad u\,|_{\Gamma_-} = 0. \qquad (*)$$

Lemma: For $f/\omega \in L^\infty(\mathcal{R}\times V)$ there exists a unique $u \in W^\infty(\omega)$ solution of (*) and

$$\|u\|_{L^\infty(\mathcal{R}\times V)} \le \|f/\omega\|\|_{L^\infty(\mathcal{R}\times V)}$$

Proof: Use $\omega = \max\{\sigma, \tau^{-1}\}$ and define

$$u(x,v) = u(x_0 + tv, v) := \int_0^t e^{-\int_s^t \sigma(x_0+rv,v)\,\mathrm{d}r} f(x_0+sv,v)\,\mathrm{d}s. \qquad (1)$$

Observe

- $v\nabla u(x,v) = \frac{d}{dt}u(x_0+tv,v) = f(x,v) - \sigma(x,v)u(x,v)$.
- $\int_0^t e^{-\int_s^t \sigma(x_0+rv,v)\,\mathrm{d}r}\omega(x_0+sv,v)\,\mathrm{d}s \le 1$.

Remark: $\|\frac{1}{\omega}v\nabla u\|_{L^\infty(\mathcal{R}\times V)} \le \|f/\omega\|_{L^\infty(\mathcal{R}\times V)} + \|u\|_{L^\infty(\mathcal{R}\times V)}$.

Contraction property

$$(v\nabla + \sigma)u = \mathcal{K}u, \qquad u\,|_{\Gamma_-} = 0.$$

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Lemma: For $u \in L^\infty(\mathcal{R} \times V)$ there holds

$$\|(v\nabla + \sigma)^{-1}\mathcal{K}u\|_{L^\infty(\mathcal{R}\times V)} \le \left(1 - e^{-\|\sigma_p'\tau\|_{L^\infty(\mathcal{R}\times V)}}\right)\|u\|_{L^\infty(\mathcal{R}\times V)},$$

where $\mathcal{K}u(x, v) := \int_V k(x, v, v')u(x, v')\,\mathrm{d}v'$.

Proof: By using $f = \mathcal{K}u$ in (1) there holds

$$\begin{aligned}|((v\nabla + \sigma)^{-1}\mathcal{K}u)(x, v)| &\le \int_0^t e^{-\int_s^t \sigma_p'(x_0+rv,v)\,\mathrm{d}r}\sigma_p'(x_0 + sv, v)\,\mathrm{d}s\ \|u\|_{L^\infty(\mathcal{R}\times V)} \\ &\le \left(1 - e^{-\|\sigma_p'\tau\|\|_{L^\infty(\mathcal{R}\times V)}}\right)\ \|u\|_{L^\infty(\mathcal{R}\times V)}.\end{aligned}$$

Solvability of RTE in L^∞

$$v\nabla u + \sigma u = \mathcal{K}u + f, \qquad u\,|_{\Gamma_-} = 0.$$

Theorem: For $f/\omega \in L^\infty(\mathcal{R} \times V)$ the RTE has a unique solution $u \in W^\infty(\omega)$ with a-priori estimate

$$\|u\|_{L^\infty(\mathcal{R}\times V)} \leq e^{\|\sigma_p'\tau\|\|_{L^\infty(\mathcal{R}\times V)}}\|f/\omega\|_{L^\infty(\mathcal{R}\times V)}$$
$$\|\frac{1}{\omega}\nabla u\|_{L^\infty(\mathcal{R}\times V)} \leq 3e^{\|\sigma_p'\tau\|\|_{L^\infty(\mathcal{R}\times V)}}\|f/\omega\|_{L^\infty(\mathcal{R}\times V)}$$

Proof: $\Phi : L^\infty(\mathcal{R} \times V) \to L^\infty(\mathcal{R} \times V)$

$$\Phi(u) := (v\nabla + \sigma)^{-1}(\mathcal{K}u + f)$$

is a contraction. Banach fixed point theorem.

Solvability in L^1: A duality argument $(L^1(\mathcal{R}\times V))' = L^\infty(\mathcal{R}\times V)$

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$$\mathcal{K}' : L^\infty(\mathcal{R}\times V) \to L^\infty(\mathcal{R}\times V),\ \mathcal{K}'u(x,v') := \int_V k(x,v,v')u(x,v)\,\mathrm{d}v$$

Lemma: For $w \in L^\infty(\mathcal{R}\times V)$ there holds

$$\|(-v\cdot\nabla+\sigma)^{-1}\mathcal{K}'w\|_{L^\infty(\mathcal{R}\times V)} \le \left(1 - e^{-\|\sigma_p\tau\|\|_{L^\infty(\mathcal{R}\times V)}}\right)\|w\|_{L^\infty(\mathcal{R}\times V)}$$

Corollary: [cf Bal, Jollivet 2008] For $u \in L^1(\mathcal{R}\times V)$ there holds

$$\|\mathcal{K}(v\nabla+\sigma)^{-1}u\|_{L^1(\mathcal{R}\times V)} \le \left(1 - e^{-\|\sigma_p\tau\|_{L^\infty(\mathcal{R}\times V)}}\right)\|u\|_{L^1(\mathcal{R}\times V)}$$

Corollary: $\Phi : L^1(\mathcal{R}\times V)\to L^1(\mathcal{R}\times V)$, $\Phi(w) := \mathcal{K}(v\nabla+\sigma)^{-1}w + f$ is a contraction.

Theorem: $u := (v\cdot\nabla+\sigma)^{-1}w \in W^1(\omega)$, with $w = \Phi(w)$, solves RTE

$$v\cdot\nabla u + \sigma u = \mathcal{K}u + f, \qquad u_{|\Gamma_-} = 0.$$

Solvability of RTE in L^p, $1 \le p \le \infty$

$$v\nabla u + \sigma u = \mathcal{K}u + f, \qquad u\,|_{\Gamma_-} = 0.$$

Theorem: For any $\frac{f}{\omega} \in L^p(\omega)$ the RTE has a unique solution $u \in W^p(\omega)$ and

$$\|u\|_{L^p(\omega)} \le C\|f/\omega\|_{L^p(\omega)}$$
$$\|\frac{1}{\omega}v \cdot \nabla u\|_{L^p(\omega)} \le 3^{1-\frac{1}{p}}C\|f/\omega\|_{L^p(\omega)}$$

with $C := \exp(\frac{1}{p}\|\sigma_p\tau\|_{L^\infty(\mathcal{R}\times V)} + (1-\frac{1}{p})\|\sigma_p'\tau\|_{L^\infty(\mathcal{R}\times V)})$ and $\omega = \{\sigma, \tau^{-1}\}$.

Proof: Interpolation.

Remark:

- In general, constants are sharp.
- If $\sigma > 0$, constants can be improved.

Conclusions

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- General L^p-solvability theory for RTE
  - Parameters are allowed to vanish
  - explicit constants
- Fixed point iteration $\equiv$ source iteration converges even if $\sigma_p = \sigma$ in all L^p
- Positive data implies positive solutions
- L^p-theory can be used to investigate continuity and differentiability of the parameter-solution map $(\sigma, k) \mapsto F(\sigma, k)$ (Hölder inequality)

Optical tomography: Reconstruction (Inverse Problem)

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1. *Given:* Measurements $(g, M) = (g, F(\sigma, k))$.
2. *Aim:* Determine $\sigma, k : F(\sigma, k) = M$.

- Uniqueness/ Identifiability: [Choulli & Stefanov]
- Stability estimates: [Bal & Jollivet]
- noisy data $\rightarrow$ ill-posedness $\rightarrow$ regularization

Example: Lipschitz-continuity $u = u(\sigma, k)$, $\tilde{u} = u(\tilde{\sigma}, k)$: $U := u - \tilde{u}$ satisfies:

$$v \cdot \nabla U + \sigma U = \mathcal{K} U + (\tilde{\sigma} - \sigma)\tilde{u}.$$

A-priori estimates

$$\|F(\sigma, k) - F(\tilde{\sigma}, k)\|_2 \leq C\|U\|_2 \leq C\|(\tilde{\sigma} - \sigma)\tilde{u}\|_2$$

- L^2 regularization: $\|(\tilde{\sigma} - \sigma)\tilde{u}\|_2 \leq \|\tilde{\sigma} - \sigma\|_2 \|\tilde{u}\|_\infty$
- H^1 regularization: $\|(\tilde{\sigma} - \sigma)\tilde{u}\|_2 \leq C\|\tilde{\sigma} - \sigma\|_{H^1} \|\tilde{u}\|_{6/5}$

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