Transmission eigenvalues for non-regular cases

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Introduction

The scattering problem for time-harmonic waves (e.g., acoustic) in a non-homogeneous medium can be modeled by the Helmholtz equation

$$\Delta u(x) + k^2(1 + m(x))u(x) = 0, \quad x \in R^n, \quad n \geq 2,$$

where $k > 0$ fixed, $m(x)$ denotes the perturbation of the index of refraction and the total wave u is equal to

$$u(x) = u_0(x) + u_{sc}(x)$$

with the incident field u_0 as the entire solution of the free Helmholtz equation and u_{sc} as the scattered field.
We assume that m is compactly supported in some domain $D \subset R^n$ and belong to $L^p(D)$ for some $\frac{n}{2} < p \leq \infty$.

Under u_0 we understand the solution of

$$(\Delta + k^2)u_0(x) = 0$$

in the form of Hergoltz function, i.e.

$$u_0(x) = \int_{S^{n-1}} e^{ik(x,\vartheta)} g_0(\vartheta)\, d\vartheta, \quad g_0 \in L^2(S^{n-1}).$$

Here as usually S^{n-1} is the unit sphere in R^n.

Introduction

It is important that for any $g_0 \in L^2(S^{n-1})$ Hergoltz function u_0 belongs to the weighted space

$$L^{\frac{2p}{p-1}}_{-\delta}(R^n),$$

where $\frac{n}{2} < p \leq \infty, n \geq 2$, and $\delta > \frac{1}{2} - \frac{n}{2p}$.
The choice of such u_0 can be justified by some works of Colton, Hörmander, Sylvester and some others. The set of all such solutions we denote by U_0.

Introduction

By U_{sc} we denote the set of all solutions of the non-homogeneous Helmholtz equation

$$\Delta u(x) + k^2(1 + m(x))u(x) = f(x)$$

with compactly supported function f which belongs to $L^{\frac{2p}{p+1}}$. This solution must satisfy the Sommerfeld radiation condition at the infinity

$$\lim_{r\to\infty} r^{\frac{n-1}{2}}(\frac{\partial u}{\partial r} - iku) = 0, \quad r = |x|.$$

By U_m we denote the set of all solutions of the Helmholtz equation in the above form such that $u_0 \in U_0$ and $u_{sc} \in U_{sc}$. The following result is true :

Theorem

For any $m \in L^p(D)$, $\frac{n}{2} < p \le \infty$, $u_0 \in U_0$, there exists a unique $u_m \in U_m$ such that

$$\|u_m\|_{L^{\frac{2p}{p-1}}_{-\delta}(R^n)} \le C\left(\|m\|_{L^p(D)} + 1\right)\|u_0\|_{L^{\frac{2p}{p-1}}_{-\delta}(R^n)}$$

We prove the following fact (which can be also considered as the basis of the approach).

Theorem

Every total wave $u_m \in U_m$ *has a unique decomposition into an incident wave* $u_0 \in U_0$ *plus a scattered wave* $u_{sc} \in U_{sc}$*, and every incident wave* $v_0 \in U_0$ *has a unique decomposition as a total wave* $v_m \in U_m$ *minus a scattered wave* $v_{sc} \in U_{sc}$ *:*

$$u_m(x) = u_0(x) + u_{sc}(x), \quad v_0(x) = v_m(x) - v_{sc}(x).$$

Formulation of the problem

As a consequence we may consider the interior transmission eigenvalue problem for "singular" m. The interior transmission eigenvalues problem is to find positive values of parameter k for which there is a non-trivial pair (u, v) solving

$$\Delta u(x) + k^2(1 + m(x))u(x) = 0, \quad x \in D,$$

$$\Delta v(x) + k^2 v(x) = 0, \quad x \in D,$$

$$u(x) = v(x), \quad \frac{\partial u(x)}{\partial \nu} = \frac{\partial v(x)}{\partial \nu}, \quad x \in \partial D.$$

This problem arises naturally in inverse scattering theory. If $k > 0$ is not a transmission eigenvalue then the far field pattern operator is injective with dense range. In that case one can apply sampling method of Colton and Kirsch and the factorization method of Kirsch and can define unknown domain D. So, the elimination of such values of k is very important in applications.

Historical remarks

The problem was first introduced in 1988 by Colton and Monk in connection with an inverse scattering problem for the reduced wave equation.

- Colton, Kirsch and Päivärinta, 1989 : the discreteness of this set.
- Rynne and Sleeman, 1991 : connection with inverse scattering theory.
- McLaughlin and Polyakov, 1994 : existence of transmission eigenvalues for the constant contrast.
- Kirsch, 1999, 2007 : Maxwell's equations.
- Colton, Päivärinta and Sylvester, 2007 : the characterization of real transmission eigenvalues.

Historical remarks

- Päivärinta and Sylvester, 2008 : the existence of transmission eigenvalues.
- Cakoni, Gintides and Haddar, 2010 : the existence of infinitely many transmission eigenvalues.
- Cakoni, Colton and Haddar, 2010 : transmission eigenvalues in presence of cavities.
- Hickmann, 2010 (is not published) : transmission eigenvalues for degenerate case.
- Hitrik, Krupchyk, Ola and Päivärinta, 2010-2011 : transmission eigenvalues for elliptic operators of arbitrary order with constant coefficients.
- .... (a lot of new publications nowadays).

Index of refraction

It is important that all these results were obtained under the hypothesis the perturbation of the index of refraction m is bounded, does not change sign and satisfies the condition $|m(x)| \geq \delta > 0$ for all x. We mentioned here one result of Hickmann (is not published) where some similar results for the degenerate case are obtained but with very restricted contrast. Unlike to this we assume that function m satisfies the following conditions :

$$c_1 \rho^{\beta} \leq m(x) \leq c_2 \rho^{\alpha}, \quad 0 < c_1 \leq c_2$$

with

$$-1 < \alpha \leq \beta < 2 + \alpha$$

where $\rho(x)$ denotes the distance from $x \in D$ to the boundary of D, that is $\rho(x) := \inf_{y \in \partial D} |x - y|$. You can see that the condition $|m(x)| \geq \delta > 0$ is not satisfied (in general). We assume also some smoothness of function $\rho(x)$ in order to apply the Hardy inequality.

Hardy inequality

Let us assume that $\sigma > 1$. Then there is a constant $C > 0$ such that for all $f \in C_0^\infty(D)$

$$\int_D \rho^{-\sigma} |f(x)|^2 \, dx \leq C \int_D \rho^{-\sigma+2} |\nabla f(x)|^2 \, dx.$$

For the proof we refer to Necas (1962) or Triebel (1980). As the consequence we obtain

$$\int_D \rho^{-\sigma+2} |\nabla f(x)|^2 \, dx \leq C \int_D \rho^{-\sigma+4} \sum_{|\gamma|=2} |\partial^\gamma f(x)|^2 \, dx,$$

where $\sigma > 3$.
These two inequalities justify the definition of the following weighted Sobolev space $H^2_{0,\beta}(D)$.

$H^2_{0,\beta}(D)$

We define the weighted Sobolev space $H^2_{0,\beta}(D)$ as the closure of $C_0^\infty(D)$ with respect to the norm

$$\|f\|^2_{H^2_{0,\beta}(D)} = \int_D \left(\rho^{-\beta} \sum_{|\gamma|=2} |\partial^\gamma f(x)|^2 + \rho^{-\beta-2} |\nabla f(x)|^2 + \rho^{-\beta-4} |f(x)|^2 \right) dx.$$

It can be easily seen that if $\beta \geq -2$ then the following embeddings hold

$$H^2_{0,\beta}(D) \subset W^1_{2,0}(D) \subset L^2(D),$$

where the latter embedding is compact. But in our further considerations we assume that $\beta > -1$ any way.

Quadratic forms

We consider the quadratic forms

$$Q_\tau(u) = Q_0(u) + 2\tau Re \int_D \overline{u} \frac{1}{m} \Delta u \, dx + \tau \int_D \overline{u} \Delta u \, dx + \tau^2 \int_D (\frac{1}{m} + 1)|u|^2 \, dx,$$

$$Q_0(u) = \int_D \frac{1}{m} |\Delta u|^2 \, dx,$$

where $\tau = k^2$, on the Hilbert space $L^2_{-\delta}(D)$. If we choose δ satisfying

$$\max(\beta - \frac{\alpha}{2}; -\frac{\alpha}{2}) \leq \delta < \frac{\alpha}{2} + 2$$

then the domain of Q_τ and Q_0 satisfy

$$H^2_{0,\beta}(D) \subset Dom(Q_\tau) = Dom(Q_0) \subset H^2_{0,\alpha}(D).$$

Characterization

Theorem

k^2 is a transmission eigenvalue of m if and only if there is a function $u \in Dom(Q_\tau), u \neq 0$, such that the following equality

$$\int_D \frac{1}{m} \left(\Delta + k^2(1+m)\right) u(x)(\Delta + k^2)\varphi(x)\, dx = 0$$

holds for any $\varphi \in Dom(Q_\tau)$.

Characterization

This theorem tells us that k^2 is a transmission eigenvalue whenever the operator

$$(\Delta + k^2)\left(\frac{1}{m}(\Delta + k^2(1+m))\right) = (\Delta + k^2(1+m))\left(\frac{1}{m}(\Delta + k^2)\right)$$

has a nontrivial kernel in $Dom(Q_\tau)$. This function u is called a transmission eigenfunction.

The mentioned above inclusions and equality for the domains follow directly from the Hardy inequality. Moreover the conditions for α and β imply that if $u \in H^2_{0,\alpha}(D)$ (as well as for $H^2_{0,\beta}(D)$) then

$$u(x) = 0, \quad \partial_\nu u(x) = 0, \quad x \in \partial D.$$

Another important property of these spaces is :
If $\delta < \frac{\alpha}{2} + 2$ then the embedding

$$H^2_{0,\alpha}(D) \subset L^2_{-\delta}(D) \subset L^2(D)$$

is compact. Actually the domain of the quadratic form Q_τ is independent on τ. It follows from the following properties.

Continuous dependence of Q_τ on τ

If we choose δ as above then for any $0 < \varepsilon < 1$ there is a constant $C_\varepsilon > 0$ such that

$$(1-\varepsilon)Q_0(u) - C_\varepsilon \|u\|^2_{L^2_{-\delta}(D)} \le Q_\tau \le (1+\varepsilon)Q_0(u) + C_\varepsilon \|u\|^2_{L^2_{-\delta}(D)}$$

and

$$|Q_{\tau_1}(u) - Q_{\tau_2}(u)| \le C|\tau_1 - \tau_2| \max(1;\tau_1;\tau_2)\left(Q_0(u) + \|u\|^2_{L^2_{-\delta}(D)}\right)$$

with some constant $C > 0$.
The second inequality illustrates the continuous dependence of Q_τ on τ. A direct consequence of these two facts is :

Eigenvalues and their dependence on τ

Theorem

If $-1 < \alpha \leq \beta < \alpha + 2$ *and* $\max(\beta - \frac{\alpha}{2}; -\frac{\alpha}{2}) \leq \delta < \frac{\alpha}{2} + 2$, *then the spectrum of self-adjoint operator* L_τ *which is understood in the sense of quadratic forms* Q_τ, *is real, discrete, of finite multiplicity and has the only one accumulation point at the infinity. Each eigenfunction* $u_l(\tau)$ *depends continuously on* τ, *belongs to* $H^2_{0,\alpha}(D)$ *and therefore must vanish, together with its derivatives, on* ∂D. *The eigenvalues can be characterized by the min-max principle*

$$\lambda_l(\tau) = \max_{V_l} \min_{u \in V_l^\perp \cap Dom(Q_\tau), \|u\|_{L^2_{-\delta}(D)}=1} Q_\tau(u),$$

where V_l *denotes any* $l-$*dimensional subspace of* $L^2_{-\delta}(D)$, $l = 1, 2, \ldots$.

We can refer to Simon.

Existence of transmission eigenvalues

The embedding

$$Dom(Q_\tau) \subset W^1_{2,0}(D),$$

where $W^1_{2,0}(D)$ denotes usual Sobolev space of functions with compact support, leads to the inequality

$$\lambda_0 \leq \inf_{u \in Dom(Q_\tau), \|u\|_{L^2(D)}=1} \int_D |\nabla u|^2 \, dx,$$

where λ_0 is the first eigenvalue of the Dirichlet Laplacian in the domain D. We introduce two constants :

$$S^+ = \sup_{u \in Dom(Q_\tau), \|u\|_{L^2(D)}=1} \int_D m|u|^2 \, dx,$$

$$S^- = \sup_{u \in Dom(Q_\tau), \|u\|_{L^2(D)}=1} \int_D \frac{1}{m}|u|^2 \, dx.$$

Existence of transmission eigenvalues

There is one conditional theorem about existence of transmission eigenvalues.

Theorem

If $k^2 < \frac{\lambda_0}{1+S^+}$ *then* k^2 *is not a transmission eigenvalue. If* $k^2 \geq \frac{\lambda_0}{1+S^+}$ *and*

$$\lambda_0 \geq 2\sqrt{\lambda_l}\left(\sqrt{1+S^-}+\sqrt{S^-}\right)$$

then there exist $l+1$ *transmission eigenvalues* k^2 *with*

$$\frac{\lambda_0 - 2\sqrt{\lambda_l}\sqrt{S^-} - \sqrt{\lambda_0^2 - 4\lambda_0\sqrt{\lambda_l}\sqrt{S^-} - 4\lambda_l}}{2(1+S^-)} \leq k^2 \leq$$

$$\leq \frac{\lambda_0 - 2\sqrt{\lambda_l}\sqrt{S^-} + \sqrt{\lambda_0^2 - 4\lambda_0\sqrt{\lambda_l}\sqrt{S^-} - 4\lambda_l}}{2(1+S^-)}.$$

Main result

Let us denote by $N_{m,D}(\tau)$ the number of all transmission eigenvalues (counting multiplicity) less than or equal to τ. Our main result is the following :

Theorem

Let D be a bounded domain in $R^n, n \geq 2$. There is a constant $K_0 > 0$, depending on both $m(x)$ and D, such that

$$N_{m,D}(\tau) \geq K_0 \tau^{\frac{n}{2}} - 1.$$

A particular consequence of this result is that there are infinitely many transmission eigenvalues.

Proof of the main result

We prove first the following proposition

Proposition

Suppose $N_{m_0,D_0}(\tau) \geq 1$, *and suppose that* $\{m_j, D_j\}_{j=1}^{K}$ *represent translation of* (m_0, D_0) *(that is,* $m_j(x) = m_0(x + x_j)$ *on* $D_j = D_0 + x_j$*). If* D_j *are disjoint, each* $D_j \subset D$ *and* $m(x) \geq m_j(x)$ *on each* D_j*, then*

$$N_{m,D}(\tau) \geq K.$$

The proof is based on the continuous dependence of eigenvalues on τ and the min-max characterization of the eigenvalues.

The next step is :
Since it is known that unit ball in R^n with any constant contrast has transmission eigenvalues (J. McLaughlin and Polyakov) then this proposition assures that the unit cube has also (at least one) transmission eigenvalues. If we denote by $\tau_0(M,1)$ the lowest transmission eigenvalue of the cube with side 1 with constant contrast M then the lowest transmission eigenvalue of the cube with side R and constant contrast M will be equal to $\frac{\tau_0(M,1)}{R^2}$. It can be seen by scaling : indeed, if $u(x)$ is a transmission eigenfunction for the cube with side length 1, then $u(\frac{x}{R})$ is a transmission eigenfunction for the cube with side length R, and the transmission eigenvalue decreases by a factor of R^2.

As an immediate consequence of this fact we obtain the following proposition.

Proposition

Suppose $m(x) > 0$ *in* D *and suppose that* $m(x) \geq M$ *on the disjoint union of* P *cubes, all with identical side length* R *and all contained in* D. *Then*

$$N_{m,D}\left(\frac{\tau_0(M,1)}{R^2}\right) \geq P.$$

Proof of the main result

Remark

If we denote by $P(R)$ *the maximum number of disjoint cubes of radius* R *contained in bounded open set* G *then*

$$R^n P(R) \to \mu(G), \quad as \quad R \to 0,$$

where $\mu(G)$ *is the Lebesgue measure of* G*.*

Proof of the main result

Proposition

Let G be an open subset of D on which $m(x) > M$ (if m is continuous then we may choose $G = \{x \in D : m(x) > M\}$). Then

$$\lim_{\tau \to +\infty} \tau^{-\frac{n}{2}} N_{m,D}(\tau) \geq (\tau_0(M,1))^{-\frac{n}{2}} \mu(G).$$

The proof follows immediately from the previous proposition and remark if we choose

$$R = \sqrt{\frac{\tau_0(M,1)}{\tau}}.$$

Proof of the main result

This proposition tells us that $N_{m,D}(\tau) + 1$ is bounded from below, for example, by

$$N_{m,D}(\tau) + 1 \geq \tau^{\frac{n}{2}} \frac{\mu(G)}{(1+\varepsilon)(\tau_0(M,1))^{\frac{n}{2}}}, \quad \textit{for} \quad \tau \geq \tau_1,$$

with some $\tau_1 > 0$ depending on $\varepsilon > 0$. It is also bounded from below by 1, for all $\tau \geq 0$. This means that the main theorem is completely proved. It must be mentioned here that the dependence of $\tau_0(M,1)$ on the constant contrast M is monotonic decreasing as well as $\mu(G)$.

Conclusions

1) It is shown the existence of real transmission eigenvalues for degenerate contrasts, which may vanish at the boundary to an arbitrary high order, but it is required some uniformity in the boundary behavior. The upper and lower bounds must involve powers of ρ (distance to the boundary) which differ by no more that two.
2) With similar restrictions, it is also allowed singular contrasts which grow at the boundary more slowly than $\frac{1}{\rho}$.
3) It is proved a lower bound of the counting function $N_{\tau,D}$, similar to that for Dirichlet eigenvalues. The number of Dirichlet eigenvalues less than or equal to λ is asymptotic to

$$\frac{\omega_n}{(2\pi)^n}\mu(D)\lambda^{\frac{n}{2}},$$

where ω_n is the volume of the unit ball in R^n. And the number of transmission eigenvalues less than or equal τ satisfies the estimate

$$N_{\tau,D} \geq K_0(m)\mu(D)\tau^{\frac{n}{2}} - 1.$$

Conclusions

4) Unfortunately, it is not known upper bound. The dependence of $\tau_0(M,1)$ on the constant contrast M in the constant

$$K_0(m) = \frac{\mu(x : m(x) > M)}{2\mu(D)(\tau_0(M,1))^{\frac{n}{2}}}$$

is monotonic decreasing (as well as $\mu(x : m(x) > M)$), but it is known little else at present.