Cordial Volterra integral equations of the first kind

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REFERENCES (The talk is based on [1–7], especially on [5,6]).

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$$\int_0^t \frac{1}{t}\varphi\left(\frac{s}{t}\right)a(t,s)u(s)ds = f(t),\ 0 < t \le T, \quad \text{or } V_{\varphi,a}u = f. \qquad (1)$$

CONTENTS

ABOUT CORDIAL VOLTERRA INTEGRAL OPERATORS

$$(V_\varphi u)(t) = \int_0^t \frac{1}{t}\varphi\big(\frac{s}{t}\big)u(s)ds, \ \ (V_{\varphi,a}u)(t) = \int_0^t \frac{1}{t}\varphi\big(\frac{s}{t}\big)a(t,s)u(s)ds, \ \ 0 < t \leq T.$$

Proposition. *Let* $\varphi \in L^1(0,1)$, $a \in C^m(\triangle_T)$ *for an* $m \geq 0$, *where* $\triangle_T = \big\{(t,s) : 0 \leq s \leq t \leq T\big\}$. *Then* $V_{\varphi,a} \in \mathcal{L}(C^m)$, *i.e.* $V_{\varphi,a}$ *is a linear* ***bounded*** *operator in* $C^m = C^m[0,T]$. *Operator* $V_{\varphi,a} \in \mathcal{L}(C^m)$ *with* $\varphi \neq 0$ *is compact if and only if* $a(0,0) = 0$. *In particular,* $V_\varphi \in \mathcal{L}(C^m)$ *for* $m = 0,1,...$ *It holds* $\| \ V_\varphi \ \|_{\mathcal{L}(C^m)} \leq \int_0^1 \mid \varphi(x) \mid dx$, $V_\varphi t^\lambda = \widehat{\varphi}(\lambda)t^\lambda$, *where* $\widehat{\varphi}(\lambda) = \int_0^1 x^\lambda \varphi(x)dx$, $\mathrm{Re}\lambda \geq 0$, *is the (shifted) Mellin transform of* φ,

$$\sigma_{\mathcal{L}(C^m)}(V_\varphi) = \{0\} \cup \{\widehat{\varphi}(k) : k = 0,1,...,m-1\} \cup \{\widehat{\varphi}(\lambda) : \mathrm{Re}\lambda \geq m\},$$

$$\sigma_{\mathcal{L}(C^m)}(V_{\varphi,a}) = a(0,0)\sigma_{\mathcal{L}(C^m)}(V_\varphi).$$

THE PROBLEM

$$\int_0^t \frac{1}{t}\varphi\Big(\frac{s}{t}\Big)a(t,s)u(s)ds = f(t),\ 0 < t \le T, \quad \text{or } V_{\varphi,a}u = f. \tag{1}$$

Assumptions: for an $r \in \mathbb{R}$ (mostly we are interested in case $r = 0$), it holds that

$$\int_0^1 x^r \mid \varphi(x) \mid dx < \infty, \tag{2}$$

$$\int_0^1 x^{r+1}(1-x) \mid \varphi'(x) \mid dx < \infty, \tag{3}$$

$$\widehat{\varphi}(r) := \int_0^1 x^r \varphi(x)dx > 0, \tag{4}$$

$$\psi_\beta(x) := \beta\varphi(x) + x\varphi'(x) \ge 0 \ (0 < x < 1) \ \text{ for a } \beta < r+1; \tag{5}$$

for an $m \ge 0$, it holds that

$$a \in C^{m+1}(\triangle_T), \ \text{ and } a(t,t) = 1 \ (0 \le t \le T). \tag{6}$$

MAIN RESULTS

Theorem 1. *Assume* (2)-(6) *for* $r = 0$. *Then* $V_{\varphi,a}$ *is injective in* C,

$$V_{\varphi,a}^{-1} \in \mathcal{L}(C^{m+1}, C^m), \quad C^{m+1} \subset V_{\varphi,a}(C^m) \subset C^m.$$

Corollary 1. *Assume* (2)-(6) *for an* $r \in \mathbb{R}$. *Then for any* f *of the form* $f(t) = t^r f_r(t)$, $f_r \in C^{m+1}$, *equation* (1) *has a unique solution* u *of a similar form* $u(t) = t^r u_r(t)$, $u_r \in C^m$. *Namely,* u_r *is a unique solution in* C^m *of the equation* $V_{\varphi_r,a} u_r = f_r$ *with* $\varphi_r(x) = \varphi(x)x^r$ $(0 < x < 1)$ *which satisfies* (2)-(6) *for* $r = 0$, hence *the inverse* $V_{\varphi_r,a}^{-1} \in \mathcal{L}(C^{m+1}, C^m)$ *exists.*

Proof sceme of Theorem 1. Rewrite (1) as

$$V_\varphi u + V_{\varphi,b} u = f, \;\; b(t,s) = a(t,s) - 1, \;\; b(t,t) \equiv 0.$$

From [5] we know that under conditions (2)-(5) for $r = 0$, V_φ is injective in C and $V_\varphi^{-1} \in \mathcal{L}(C^{m+1}, C^m)$. We prove that $V_{\varphi,b} \in \mathcal{L}(C^m, C^{m+1})$ and that this operator is compact if $\partial b/\partial t \mid_{t=s=0} = 0$. Thus equation (1) is equivalent to the second kind equation

$$u + V_\varphi^{-1} V_{\varphi,b} u = V_\varphi^{-1} f.$$

The compactness of $V_\varphi^{-1}V_{\varphi,b} \in \mathcal{L}(C^m)$ can be achieved by treating (1) w.r.t. new unknown $\overline{u}(t) = \mathrm{e}^{-\mu t}u(t)$ with $\mu = \partial a(t,s)/\partial t\big|_{t=s=0}$; this changes $b(t,s)$ into $\overline{b}(t,s) = a(t,s)\mathrm{e}^{-\mu(t-s)}$. Now $\partial\overline{b}/\partial t\ |_{t=s=0} = 0$, and $V_{\varphi,\bar{b}} \in \mathcal{L}(C^m, C^{m+1})$ is compact.

We show that the homogeneous equation $u + V_\varphi^{-1}V_{\varphi,b}u = 0$ has in C only the trivial solution. The claims of the Theorem follow by the Fredholm alternative. $\square$

Introduce the weighted space $C_\star^{m,r} = C_\star^{m,r}(0,T]$ consisting of functions $u \in C^m(0,T]$ such that finite limits $\lim_{t\to 0} t^{k-r}u^{(k)}(t)$, $k = 0, ..., m$, exist; the norm in $C_\star^{m,r}$ is defined by

$$\| u \|_{C_\star^{m,r}} = \max_{0\le k\le m} \sup_{0<t\le T} t^{k-r} \mid u^{(k)}(t) \mid .$$

Theorem 2. *Assume* (2)-(6) *for an* $r \in \mathbb{R}$. *Then* $V_{\varphi,a}$ *is injective in* $C_\star^{0,r}$,

$$V_{\varphi,a}^{-1} \in \mathcal{L}(C_\star^{m+1,r}, C_\star^{m,r}), \quad C_\star^{m+1,r} \subset V_{\varphi,a}(C_\star^{m,r}) \subset C_\star^{m,r}.$$

Also for Hoelder spaces, a counterpart of Theorem 1 can be established.

APPLICATION TO ABEL TYPE EQUATION

$$\int_0^t (t^\gamma - s^\gamma)^{-\nu} g(s/t) a(t,s) u(s) ds = t^\lambda f(t), \tag{7}$$

where γ,ν,λ are real parameters,

$$\gamma > 0,\ \ 0 < \nu < 1,\ \ \lambda + \gamma\nu > 0,\ \ 0 \neq g \in W^{1,\infty}(0,1),\ \ g \geq 0,\ \ g' \geq 0. \tag{8}$$

With respect to $u_\beta(t) = t^\beta u(t)$, $\beta = 1 - \lambda - \gamma\nu$, equation (9) can be represented in the form of cordial equation (1) with the core function

$$\varphi(x) = x^{-\beta}(1 - x^\gamma)^{-\nu} g(x),\ \ \beta = 1 - \lambda - \gamma\nu < 1, \text{ satisfying (2)-(5) for } r = 0.$$

Applying Theorem 1 we obtain

Theorem 3. *Assume* (8) *and* $a \in C^{m+1}(\triangle_T)$ *for an* $m \geq 0$, $a(t,t) \neq 0$ $(0 \leq t \leq T)$. *Then for any* $f \in C^{m+1}$ *equation* (7) *has a unique solution of the form* $u(t) = t^{-\beta} u_\beta(t)$, $\beta = 1 - \lambda - \gamma\nu < 1$, $u_\beta \in C^m$, $\| u_\beta \|_{C^m} \leq c \| f \|_{C^{m+1}}$.

In case $g \equiv 1$ equation (7) has been studied by Atkinson [9]; then Theorem 3 slightly strengthens the result of Atkinson: instead of $a \in C^{m+1}(\triangle_T)$, in [9] it is assumed that $a \in C^{m+2}(\triangle_T)$.

STABILITY OF THE SOLUTION

Consider the perturbed equation $V_{\varphi,\tilde{a}}u = \tilde{f}$.

Theorem 4. *Let* φ *satisfy* (2)-(5) *for* $r = 0$, $a \in C^{m+1}(\triangle_T)$ *for an* $m \geq 0$, *and* $a(t,t) \neq 0$ *for* $0 \leq t \leq T$, *whereas* $\tilde{a} \in C^{k+1}(\triangle_T)$ *for a* k, $0 \leq k \leq m$. *Then there exists a* $\delta_0 > 0$ *such that condition*

$$\| \tilde{a} - a \|_{C^{k+1}(\triangle_T)} \leq \delta_0$$

implies the existence of a unique solution $\tilde{u} = V_{\varphi,\tilde{a}}^{-1}\tilde{f} \in C^k$ *of the perturbed equation* $V_{\varphi,\tilde{a}}u = \tilde{f}$ *for any* $\tilde{f} \in C^{k+1}$. *For* $u = V_{\varphi,a}^{-1}f \in C^m$, *the stability estimate*

$$\| \tilde{u} - u \|_{C^k} \leq c \left(\| \tilde{a} - a \|_{C^{k+1}(\triangle_T)} + \| \tilde{f} - f \|_{C^{k+1}} \right)$$

holds with a constant c *independent of perturbed data* $\tilde{a}$ *and* $\tilde{f}$.

For a perturbation of φ, the result is different (Theorems 5 and 6).

Theorem 5. *Assume* (2)-(6) *for* $r = 0$. *Then there is a* $\delta_0 > 0$ *such that for any* $\tilde{\varphi}$ *satisfying* $\beta\tilde{\varphi}(x) + x\tilde{\varphi}'(x) \geq 0$ $(0 < x < 1)$ *with a* $\beta < 1$ *independent of* $\tilde{\varphi}$ (cf. (5)), *condition*

$$\delta := \int_0^1 \big(\mid \tilde{\varphi}(x) - \varphi(x) \mid + x(1-x) \mid \tilde{\varphi}'(x) - \varphi'(x) \mid\big) dx \leq \delta_0$$

implies the existence of the inverses $V_{\tilde{\varphi},a}^{-1} \in \mathcal{L}(C^{k+1}, C^k)$, *and*

$$\| V_{\tilde{\varphi},a}^{-1} f - V_{\varphi,a}^{-1} f \|_{C^k} \to 0 \text{ as } \delta \to 0, \ \ \forall f \in C^{k+1}, \ k = 0, ..., m,$$

$$\| V_{\tilde{\varphi},a}^{-1} - V_{\varphi,a}^{-1} \|_{\mathcal{L}(C^{k+1}, C^{k-1})} \leq c\delta, \ k = 1, ..., m, \ \text{ with } c \text{ independent of } \tilde{\varphi}.$$

Let us concretise the last theorem for the case $\tilde{\varphi} = \varphi_\varepsilon$,

$$\varphi_\varepsilon(x) = \begin{cases} \varphi(x), & 0 < x \le 1-\varepsilon \\ \varphi(1-\varepsilon)(1-\varepsilon)^\beta x^{-\beta}, & 1-\varepsilon < x \le 1 \end{cases}, \quad 0 < \varepsilon \le \varepsilon_0. \tag{9}$$

Theorem 6. *Under conditions* (2)-(6),

$$\| V^{-1}_{\varphi_\varepsilon,a} f - V^{-1}_{\varphi,a} f \|_{C^k} \to 0 \ as\ \varepsilon \to 0, \ \forall f \in C^{k+1}, \ 0 \le k \le m,$$

$$\| V^{-1}_{\varphi_\varepsilon,a} - V^{-1}_{\varphi,a} \|_{\mathcal{L}(C^{k+1},C^{k-1})} \le c\delta_\varepsilon, \ \ 1 \le k \le m,$$

where

$$\delta_\varepsilon := \int_0^1 \big(| \varphi(x) - \varphi_\varepsilon(x) | + x(1-x) | \tilde{\varphi}'(x) - \varphi'_\varepsilon(x) |\big) dx$$

$$\le (1 + (| \beta | + \beta)\varepsilon) \int_{1-\varepsilon}^1 \varphi(x) dx + \int_{1-\varepsilon}^1 x(1-x)\varphi'(x) dx.$$

POLYNOMIAL COLLOCATION

Denote by $\Pi_n = \Pi_{n,[0,T]}$ the Chebyshev interpolation projector to the space $\mathcal{P}_n$ of polynomials of degree $\leq n$: for $v \in C$, $\Pi_n v \in \mathcal{P}_n$, $(\Pi_n v)(t_i) = v(t_i)$, $i = 0, ..., n$, where $t_i = \frac{T}{2}\left(1 + \cos\frac{2i+1}{2(n+1)}\pi\right)$ are the Chebyshev knots in the interval $[0, T]$. Consider the solving of equation (1) by the collocation method

$$u_n \in \mathcal{P}_n, \;\; \Pi_n V_{\varphi,a} u_n = \Pi_n f.$$

Theorem 7. *Assume that* φ satisfies (4)-(5) *for* $r = 0$ *and, for a* $\nu \in [0, 1)$,

$$\mid x^k(1-x)^k\varphi^{(k)}(x) \mid \leq cx^{-\nu}(1-x)^{-\nu} \;\; (0 < x < 1), \;\; k = 0, 1, 2,$$

whereas a *satisfies* (6) *for an* $m \geq 1$, *and* $\partial a(t,s)/\partial t \;|_{t=s=0} = 0$. *Then for any* $f \in C^{m+1}$ *equation* (1) *has a unique solution* $u \in C^m$, *for sufficiently big* n *there is a unique collocation solution* $u_n \in \mathcal{P}_n$, *and with* c *independent of* n, T *and* f,

$$\| u - u_n \|_C \leq cT^m n^{-m}(1 + \log n) \| u^{(m)} \|_C \; .$$

Using an idea of [2] the method can be reorganised into a discrete version of complexity $O(n^3)$ flops and accuracy $\| u - u_n \|_C \leq cT^m n^{-m+1}(1 + \log n)^2 \| u \|_{C^m}$. Notice a high accuracy of methods if T is small.

SPLINE COLLOCATION

Consider uniformly spaced grid points ih, $i = 0, ..., N$, $h = T/N$, splines (in general discontinuous at ih, $i = 1, ..., N-1$) of degree $n-1$, $n \geq 1$, and some collocation (interpolation) parameters $\tau_1, ..., \tau_n$, $0 \leq \tau_1 < \tau_2 < ... < \tau_n \leq 1$. Introduce corresponding spline interpolation projector P_N in a usual way.

1. Solving equation $u + V_\varphi^{-1} V_{\varphi,b} u = V_\varphi^{-1} f$. It is easy to justify the spline collocation method $u_N + P_N V_\varphi^{-1} V_{\varphi,b} u_N = P_N V_\varphi^{-1} f$ since we may assume the compactness of $V_\varphi^{-1} V_{\varphi,b} \in \mathcal{L}(C)$. But numerical realisation of the method is usually too complicated because of factor V_φ^{-1}. Neverteless, if V_φ^{-1} has a representation

$$V_\varphi^{-1} = V_\psi(D_\star - \beta I) \text{ for a } \psi \in L^1(0,1), \text{ where } (D_\star u)(t) = (tu(t))', \qquad (10)$$

then, with ψ in the hand, discrete versions of spline collocation are easily realisable; moreover, (11) implies the compactness of $V_\varphi^{-1} \in \mathcal{L}(C^{m+1}, C^m)$. For instance, for $\varphi(x) = x^{-\beta}(1 - x^\gamma)^{-\nu}$, $\beta < 1$, $\gamma > 0$, $0 < \nu < 1$, representation (10) holds with $\psi(x) = \gamma \frac{\sin(\pi\nu)}{\pi} x^{-\beta+\gamma(1-\nu)}(1 - x^\gamma)^{\nu-1}$.

A possibility to represent V_φ^{-1} in form (10) is an open problem, in general.

2. Solving a second kind version of (1) free of V_φ^{-1}. Consider first the case where (2)-(6) for $r=0$ are accomplished by condition $\int_0^1 x \mid \varphi'(x) \mid dx < \infty$; then a finite limit $\lim_{x\to 1}\varphi(x) := \varphi(1) > 0$ exists. Applying $D_\star - \beta I$ to both sides of equation $V_\varphi u + V_{\varphi,b} u = f$, rewrite (1) in an equivalent form

$$\varphi(1)u = V_{\psi_\beta}u + V_{\psi_\beta,b}u - V_{\varphi,b_1}u + (D_\star - \beta I)f \tag{11}$$

with $\psi_\beta \geq 0$ introduced in (5) and $b_1(t,s) = t\,\partial b(t,s)/\partial t$. Operators $V_{\psi_\beta,b}$ and V_{φ,b_1} are compact in C, operator V_{ψ_β} is noncompact. To the second kind cordial Volterra integral equation (11) one can apply the results [3,4,7] about the convergence and optimal convergence speed of spline collocation methods and their discrete versions, as well as about the matrix form of those. We omit detailed reformulations (those are quite straightforward) but we recall that a special "applicability condition" is necessary. Namely, for the simplified equation $\varphi(1)u = V_{\psi_\beta}u + f$, the unique solvability of the spline collocation system corresponding to first subintervals $[ih, (i+1)h]$, $i = 1, ..., i_0$, must be either assumed or, if possible, established; here i_0 is sufficiently big but independent of N, see [3,4,7] for details.

3. Case $\int_0^1 x \mid \varphi'(x) \mid dx = \infty$. Assuming still (2)-(6) for $r = 0$, we propose to approximate φ by φ_ε defined in (9) and to solve the regularized equation $V_{\varphi_\epsilon,a}u = f$ equivalent to (cf. (11))

$$\varphi_\varepsilon(1)u = V_{\psi_{\beta;\varepsilon}}u + V_{\psi_{\beta;\epsilon},b}u - V_{\varphi_\varepsilon,b_1}u + (D_\star - \beta I)f; \tag{12}$$

here $\psi_{\beta;\epsilon}(x) := \beta\varphi_\varepsilon(x) + x\varphi_\varepsilon'(x)$. Note that φ_ε inherits from φ properties (2)-(6), $r = 0$, but $\int_0^1 x \mid \varphi_\varepsilon'(x) \mid dx < \infty$. Theorem 4 enables to control the error caused by the approximation of φ by φ_ε.

4. A hybrid polynomial/spline collocation method: on $[0, T_0]$ solve (1) by polynomial collocation of degree n and continue on $[T_0, T]$ by spline collocation of degree $m - 1$ for equation (11) or (12). Taking $T_0 \in (0, T)$ small, say, of order 0.1, a prescribed accuracy of polynomial collocation on $[0, T_0]$ can be achieved for a relatively small n (see Theorem 7). For sufficiently small h, the applicability condition of spline collocation on $[T_0, T]$ is fulfilled. By this strategy one can overcome the difficulties caused by the noncompactness of operators V_{ψ_β} and $V_{\psi_{\beta;\varepsilon}}$.

THANK YOU VERY MUCH FOR YOUR ATTENTION