Free Probability approach to Random Matrices An alternative Cumulant–Moment relation formula

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Outline

80's Dan Voiculescu

- analysis of some special classes of von Neumann algebra

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Then the free moments of a self–adjoint element $a \in \mathcal{A}$ are defined as

$$m_k := \tau(a^k) := \int_{\mathbb{R}} x^k d\mu(x)$$

and they characterize a compactly supported $*$–distribution of a. The $*$–distribution is denoted by μ and $supp(\mu) \subset \mathbb{R}$.

for Random Matrices - space $(RM_p(\mathbb{C}), \tau)$

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Let $(\Omega, \mathcal{F}, P)$ be a probability space. The $RM_p(\mathbb{C})$ denotes set of all $p \times p$ matrices, with entries which belongs to $\bigcap_{p=1,2,\ldots} L^p(\Omega, P)$. Defined in this way set is a $*$-algebra, with matrix multiplication as product and conjugate transpose as $*$-operation. The $*$-algebra is equipped in trace functional τ as

$$\tau(X) := \mathbb{E}(\mathrm{Tr}_p(X)) = \mathbb{E}\left(\frac{1}{p}\mathrm{Tr}(X)\right) = \frac{1}{p}\mathbb{E}(\sum_{i=1}^{p} X_{ii}) = \frac{1}{p}\sum_{i=1}^{p}\mathbb{E}\lambda_i,$$

where $X = (X_{ij})_{i,j=1}^{p} \in RM_p(\mathbb{C})$.

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Recursive + non–crossing partitions

Let $(\mathcal{A}, \tau)$ be a non-commutative probability space. Then we define the *cumulant* functionals $k_k : \mathcal{A}^k \to \mathbb{C}$, for all $i \in \mathbb{N}$ by the moment-cumulant relation

$$k_1(a) = \tau(a), \qquad \tau(a_1 \cdots a_k) = \sum_{\pi \in NC(k)} k_\pi[a_1, \ldots, a_k],$$

where the sum is taken over all non-crossing partitions of the set $\{a_1, a_2, \ldots, a_k\}$, where $a_i \in \mathcal{A}$ for all $i = 1, 2, \ldots, k$ and

$$\begin{aligned} k_\pi[a_1, \ldots, a_k] &= \prod_{i=1}^{r} k_{V(i)}[a_1, \ldots, a_k] \qquad \pi = \{V(1), \ldots, V(r)\}, \\ k_V[a_1, \ldots, a_k] &= k_s(a_{v(1)}, \ldots, a_{v(s)}) \qquad V = (v(1), \ldots, v(s)). \end{aligned}$$

Non–crossing partitions =

Number of n–c partitions of $\{1, 2, \ldots, n\}$
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Recursive + non–crossing partitions

$$k_\pi[a_1,\ldots,a_k] = \sum_{\sigma\in NC(k),\sigma\leq\pi} \tau_\sigma[a_1,\ldots,a_k]\mu(\sigma,\pi),$$

where
$\tau_k(a_1,\ldots,a_k) := \tau(a_1\cdots a_k)$, $\tau_\pi[a_1,\ldots,a_k] := \prod_{V\in\pi}\tau_V[a_1,\ldots,a_k]$,
$\tau_V[a_1,\ldots,a_k] := \tau_k(a_{i_1},\ldots,a_{i_k})$ for $V = \{(i_1,\ldots,i_k) : i_1 < \ldots < i_k\}$
and μ is the Möbius function on $NC(k)$.

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Non–recursive cumulant–moment formula - n–c partitions

$$
\begin{aligned}
k_p &= m_p + \sum_{j=2}^{p} \frac{(-1)^{j-1}}{j} \binom{p+j-2}{j-1} \sum_{Q_j} m_{q_1} \cdots m_{q_j}, \\
m_p &= k_p + \sum_{j=2}^{p} \frac{1}{j} \binom{p}{j-1} \sum_{Q_j} k_{q_1} \cdots k_{q_j},
\end{aligned}
$$

where $Q_j = \{(q_1, q_2, \ldots, q_j) \in \mathbb{N}^j \,|\, \sum_{i=1}^{j} q_i = p\}$.

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On Voiculescu's R- and S-Transforms for free Non–Commuting Random Variables.
Fields Institute Communications, Vol. 12

Recursive - non–crossing partitions

Notation:

$$\binom{\mathbf{m}, h, \succ}{t} = \sum_{\substack{i_1+i_2+\ldots+i_h=t \\ \forall_k i_k \succ 0}} m_{i_1} m_{i_2} \cdot \ldots \cdot m_{i_h},$$

where m_i denotes ith moment and $\succ$ reflect the ordering relation.

Theorem

Let $\{k_i\}_{i=1}^{\infty}$ be the free cumulants and $\{m_i\}_{i=1}^{\infty}$ be the free moments for an element of a non–commutative probability space. Then, the following recursive formula holds $k_1 = m_1$ and for $t = 2, 3, \ldots$

$$k_t = \sum_{i=1}^{t} (-1)^{i+1} \binom{\mathbf{m}, i, >}{t} - \sum_{h=2}^{t-1} k_h \binom{\mathbf{m}, h-1, \geq}{t-h}.$$

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Definition (Stieltjes transform)

Let μ be a non-negative, finite borel measure on the $\mathbb{R}$. Then we define the Stieltjes transform of μ by the formula

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x} d\mu(x),$$

for all $z \in \{z : z \in \mathbb{C}, \Im(z) > 0\}$, where $\Im(z)$ denotes imaginary part of the complex z.

Theorem (Stieltjes inversion formula)

For any open interval $I = (a, b)$, such that neither a nor b are atoms for the probability measure μ the inversion formula

$$\mu(I) = -\frac{1}{\pi} \lim_{y \to 0} \int_I \Im G_\mu(x + \mathbf{i}y) dx$$

holds.
Here convergence is with respect to the weak topology on the space of all real probability measures.

Theorem

Let the free moments $m_k = \int_{\mathbb{R}} x^k d\mu(x)$, $k = 1, 2, \ldots$. Then

$$G_\mu(z) = \frac{1}{z}\left(1 + \sum_{i=1}^{\infty} z^{-i} m_i\right).$$

Sketch of the proof

Firstly

$$
\begin{aligned}
z &= G_\mu^{-1}(G_\mu(z)) \\
&= z + z\sum_{j=1}^{\infty}\left(-\sum_{i=1}^{\infty} z^{-i} m_i\right)^j + \sum_{i=0}^{\infty}\frac{k_{i+1}}{z^i}\left(\sum_{j=0}^{\infty} z^{-j} m_j\right)^i.
\end{aligned}
$$

Then

$$
z\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\binom{j+1}{l}(-1)^{l+1}\left(\sum_{i=0}^{\infty} z^{-i} m_i\right)^l = \sum_{i=0}^{\infty}\frac{k_{i+1}}{z^i}\left(\sum_{j=0}^{\infty} z^{-j} m_j\right)^i.
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By

$$\left(\sum_{i=0}^{\infty} m_i z^i\right)^k = \sum_{n=0}^{\infty} \binom{\mathbf{m}, k}{n} z^n,$$

we have

$$\sum_{j=0}^{\infty}\left(-1+\sum_{l=1}^{j+1}\binom{j+1}{l}(-1)^{l+1}\sum_{t=0}^{\infty}\binom{\mathbf{m}, l}{t} z^{-t}\right)$$
$$= \frac{k_1}{z} + \sum_{i=1}^{\infty} k_{i+1} \sum_{t=0}^{\infty} \binom{\mathbf{m}, i}{t} z^{-(t+i+1)}.$$

By the identification of coefficients of z^{-t} and inductive proof we obtain

$$k_t = \sum_{j=0}^{\infty} \sum_{l=1}^{j+1} \binom{j+1}{l} (-1)^{l+1} \binom{\mathbf{m}, l, \geq}{t} - \sum_{i=1}^{t-2} k_{i+1} \binom{\mathbf{m}, i, \geq}{t-i-1}.$$

Then we show that

$$\sum_{j=t}^{\infty} \sum_{l=1}^{j+1} \binom{j+1}{l} (-1)^{l+1} \binom{\mathbf{m}, l, \geq}{t} = 0$$

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\begin{aligned}
k_5 &= \sum_{i=1}^{5} (-1)^{i+1} \sum_{\substack{j_1+\ldots+j_i=5 \\ \forall_k j_k > 0}} m_{j_1} \cdot \ldots \cdot m_{j_i} \\
&\quad - \sum_{h=2}^{4} k_h \sum_{\substack{j_1+\ldots+j_{h-1}=5-h \\ \forall_k j_k \geq 0}} m_{j_1} \cdot \ldots \cdot m_{j_{h-1}}
\end{aligned}
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&= m_5-2m_1m_4-2m_3m_2+3m_1^2m_3+3m_1m_2^2-4m_1^3m_2+m_1^5\\
&\quad -k_2m_3-k_3(2m_2+m_1^2)-3k_4m_1
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&= m_5-2m_1m_4-2m_3m_2+3m_1^2m_3+3m_1m_2^2-4m_1^3m_2+m_1^5\\
&\quad -k_2m_3-k_3(2m_2+m_1^2)-3k_4m_1\\
&= m_5-5m_4m_1-5m_3m_2+15m_3m_1^2+15m_2^2m_1-35m_2m_1^3+14m_1^5
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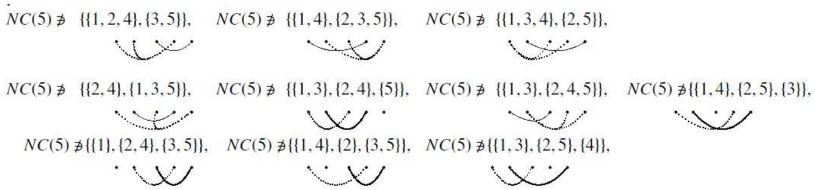
$$



$NC(5) \not\ni \{\{1,2,4\},\{3,5\}\}$, $NC(5) \not\ni \{\{1,4\},\{2,3,5\}\}$, $NC(5) \not\ni \{\{1,3,4\},\{2,5\}\}$,

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$$\begin{aligned} m_5 &= \sum_{\pi \in NC(5)} k_\pi[a,a,a,a,a] \\ &= k_5 + 5k_4k_1 + \left(\binom{5}{2} - 5\right)k_3k_2 + \binom{5}{3}k_3k_1^2 \\ &\quad + \left(\binom{5}{1}\frac{1}{2}\binom{4}{2} - 5\right)k_2^2k_1 + \binom{5}{2}k_2k_1^3 + k_1^5 \\ &= k_5 + 5k_4k_1 + 5k_3k_2 + 10k_3k_1^2 + 10k_2^2k_1 + 10k_2k_1^3 + k_1^5, \end{aligned}$$

$$k_5 = m_5 - 5m_4m_1 + 15m_3m_1^2 + 15m_2^2m_1 - 35m_2m_1^3 - 5m_3m_2 + 14m_1^5.$$

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Thank you for your attention!