Hypothesis testing in variance components with constraints

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For LMM with one variance component, Crainiceanu and Ruppert (2004) had shown that the distribution of the LR statistic is mixture of two chi-squares.

This is due to the fact that the values of the variance component under the null lie on the boundry of the parameter space.
Introduction

Aims

1. Construct statistical tests for variance components with positivity constraints.
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1. Construct statistical tests for variance components with positivity constraints.
2. Investigate the performance of the proposed procedures.
One-way model
Statistical model

\[ y = \mu 1 + Z\alpha + e, \]

where

- \( y \) - vector of observed values,
- \( \mu \) - general mean
- \( 1_N \) - vector of one’s
- \( \alpha \) - vector of random effects,
- \( Z = I_a \otimes 1_n \) is design matrix,
- \( e \) - vector of errors

We assume that \( \alpha \) and \( e \) are independent and that \( \alpha \sim N(0, \sigma^2_a I_a) \) and \( e \sim N(0, \sigma^2_e I_N) \), respectively.
One-way model
Statistical model

The problem of interest is:

\[ H_0 : \sigma_a^2 = 0 \quad \text{vs.} \quad H_1 : \sigma_a^2 > 0. \]
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Alternatively, the problem of interest can be written as:

\[ H'_0 : \gamma_1 = \gamma_2 \quad \text{vs.} \quad H'_1 : \gamma_1 > \gamma_2, \]
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Alternatively, the problem of interest can be written as:

\[ H'_0 : \gamma_1 = \gamma_2 \quad \text{vs.} \quad H'_1 : \gamma_1 > \gamma_2, \]

where \( \gamma_1 = n\sigma_a^2 + \sigma_e^2 \) and \( \gamma_2 = \sigma_e^2 \) (see Khuri et al., 1998).
The maximum likelihood estimators (MLE)

\[ \hat{\mu} = \frac{1}{N} 1_N' y, \quad \hat{\gamma}_1 = \frac{1}{a-1} y' P_1 y \quad \hat{\gamma}_2 = \frac{1}{N-a} y' P_2 y, \]

where

\[ P_1 = I_a \otimes J_n - \frac{J_N}{N} \quad P_2 = I_a \otimes I_n - \frac{I_a \otimes J_n}{n}. \]
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\( E(\hat{\gamma}_1) = \gamma_1 \) and \( V(\hat{\gamma}_1) \approx \frac{2\gamma_1^2}{a - 1}; \)
One-way model
Statistical model

- The maximum likelihood estimators (MLE)

$$\hat{\mu} = \frac{1}{N} 1_N'y,$$
$$\hat{\gamma}_1 = \frac{1}{a - 1} y'P_1y,$$
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where
$$P_1 = I_a \otimes J_n - \frac{J_N}{N},$$
$$P_2 = I_a \otimes I_n - \frac{I_a \otimes J_n}{n}.$$

- $E(\hat{\gamma}_1) = \gamma_1$ and $V(\hat{\gamma}_1) \approx \frac{2\gamma_1^2}{a - 1};$ $E(\hat{\gamma}_2) = \gamma_2$ and; $V(\hat{\gamma}_2) \approx \frac{2\gamma_2^2}{N - a}.$
One-way model
Statistical model

- The maximum likelihood estimators (MLE)

\[ \hat{\mu} = \frac{1}{N} \mathbf{1}' \mathbf{y}, \quad \hat{\gamma}_1 = \frac{1}{a-1} \mathbf{y}' \mathbf{P}_1 \mathbf{y} \quad \hat{\gamma}_2 = \frac{1}{N-a} \mathbf{y}' \mathbf{P}_2 \mathbf{y}, \]

where

\[ \mathbf{P}_1 = \frac{\mathbf{I}_a \otimes \mathbf{J}_n}{n} - \frac{\mathbf{J}_N}{N} \quad \mathbf{P}_2 = \mathbf{I}_a \otimes \mathbf{I}_n - \frac{\mathbf{I}_a \otimes \mathbf{J}_n}{n}. \]

- \( E(\hat{\gamma}_1) = \gamma_1 \) and \( V(\hat{\gamma}_1) \approx \frac{2\gamma_1^2}{a-1} \); \( E(\hat{\gamma}_2) = \gamma_2 \) and; \( V(\hat{\gamma}_2) \approx \frac{2\gamma_2^2}{N-a} \).

- By the independence of \( \hat{\gamma}_1 \) and \( \hat{\gamma}_2 \) (see Theorem 2.4.1, Khuri et al., 1998) we have
The maximum likelihood estimators (MLE)

\[ \hat{\mu} = \frac{1}{N} \mathbf{1}'_N \mathbf{y}, \quad \hat{\gamma}_1 = \frac{1}{a - 1} \mathbf{y}' \mathbf{P}_1 \mathbf{y} \quad \hat{\gamma}_2 = \frac{1}{N - a} \mathbf{y}' \mathbf{P}_2 \mathbf{y}, \]

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\[ E(\hat{\gamma}_1) = \gamma_1 \text{ and } V(\hat{\gamma}_1) \approx \frac{2\gamma_1^2}{a - 1}; \quad E(\hat{\gamma}_2) = \gamma_2 \text{ and; } V(\hat{\gamma}_2) \approx \frac{2\gamma_2^2}{N - a}. \]

By the independence of \( \hat{\gamma}_1 \) and \( \hat{\gamma}_2 \) (see Theorem 2.4.1, Khuri et al., 1998) we have

\[ \mathbf{V}_{\hat{\gamma}} = diag \left( \frac{2\gamma_1^2}{a - 1}, \frac{2\gamma_2^2}{N - a} \right). \]
The maximum likelihood estimators (MLE)

\[ \hat{\mu} = \frac{1}{N} 1'_N y, \quad \hat{\gamma}_1 = \frac{1}{a - 1} y' P_1 y, \quad \hat{\gamma}_2 = \frac{1}{N - a} y' P_2 y, \]

where

\[ P_1 = \frac{I_a \otimes J_n}{n} - \frac{J_N}{N}, \quad P_2 = I_a \otimes I_n - \frac{I_a \otimes J_n}{n}. \]

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By the independence of \( \hat{\gamma}_1 \) and \( \hat{\gamma}_2 \) (see Theorem 2.4.1, Khuri et al., 1998) we have

\[ V_{\hat{\gamma}} = \text{diag} \left( \frac{2\gamma_1^2}{a - 1}, \frac{2\gamma_2^2}{N - a} \right) \]

\[ V_{\hat{\gamma}}^{-1/2} (\hat{\gamma} - \gamma) \xrightarrow{d} N(0, I_2). \]
One-way model
Likelihood ratio

- Using the Self and Liang approach (Self and Liang, 1987) and (1)
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\[ \ell_1 = - \frac{a - 1}{2} \left( \frac{\hat{\gamma}_1}{\gamma_1} - 1 \right)^2 - \frac{N - a}{2} \left( \frac{\hat{\gamma}_2}{\gamma_2} - 1 \right)^2 \]
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- In case \( \gamma_1 > \gamma_2 \)
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- In case \( \gamma_1 > \gamma_2 \Rightarrow \tilde{\gamma}_1 = \hat{\gamma}_1 \text{ and } \tilde{\gamma}_2 = \hat{\gamma}_2 \)
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Likelihood ratio

- Using the Self and Liang approach (Self and Liang, 1987) and (1)

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- In case \( \gamma_1 > \gamma_2 \Rightarrow \tilde{\gamma}_1 = \widehat{\gamma}_1 \) and \( \tilde{\gamma}_2 = \widehat{\gamma}_2 \Rightarrow \ell_{11} = 0. \)
One-way model
Likelihood ratio

- Using the Self and Liang approach (Self and Liang, 1987) and (1)

\[
\ell_1 = -\frac{a-1}{2} \left( \frac{\hat{\gamma}_1 - 1}{\gamma_1} \right)^2 - \frac{N-a}{2} \left( \frac{\hat{\gamma}_2 - 1}{\gamma_2} \right)^2
\]

- In case \( \gamma_1 > \gamma_2 \) ⇒ \( \tilde{\gamma}_1 = \hat{\gamma}_1 \) and \( \tilde{\gamma}_2 = \hat{\gamma}_2 \) ⇒ \( \ell_{11} = 0 \).
- In case \( \gamma_1 \leq \gamma_2 \)
One-way model
Likelihood ratio

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Likelihood ratio

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In case \( \gamma_1 \leq \gamma_2 \Rightarrow \gamma_1 = \gamma_2 = \gamma \)

\[ \ell_{12} = -\frac{a - 1}{2} \left( \frac{\hat{\gamma}_1}{\gamma} - 1 \right)^2 - \frac{N - a}{2} \left( \frac{\hat{\gamma}_2}{\gamma} - 1 \right)^2 \]
One-way model
Likelihood ratio

- Using the Self and Liang approach (Self and Liang, 1987) and (1)

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\ell_1 = -\frac{a - 1}{2} \left(\frac{\hat{\gamma}_1}{\gamma_1} - 1\right)^2 - \frac{N - a}{2} \left(\frac{\hat{\gamma}_2}{\gamma_2} - 1\right)^2
\]

- In case \(\gamma_1 > \gamma_2\) ⇒ \(\tilde{\gamma}_1 = \hat{\gamma}_1\) and \(\tilde{\gamma}_2 = \hat{\gamma}_2\) ⇒ \(\ell_{11} = 0\).

- In case \(\gamma_1 \leq \gamma_2\) ⇒ \(\gamma_1 = \gamma_2 = \gamma\)

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\ell_{12} = -\frac{a - 1}{2} \left(\frac{\hat{\gamma}_1}{\gamma} - 1\right)^2 - \frac{N - a}{2} \left(\frac{\hat{\gamma}_2}{\gamma} - 1\right)^2
\]

\[
\frac{d\ell_{12}}{d\gamma^{-1}} = -\hat{\gamma}_1 (a - 1) \left(\frac{\hat{\gamma}_1}{\gamma} - 1\right) - \hat{\gamma}_2 (N - a) \left(\frac{\hat{\gamma}_2}{\gamma} - 1\right) = 0.
\]
One-way model
Likelihood ratio

\[ \gamma_1 \leq \gamma_2 \Rightarrow \]
One-way model
Likelihood ratio

\( \gamma_1 \leq \gamma_2 \Rightarrow \)

\[ \tilde{\gamma} = \frac{(a - 1) \hat{\gamma}_1^2 + (N - a) \hat{\gamma}_2^2}{(a - 1) \hat{\gamma}_1 + (N - a) \hat{\gamma}_2}, \quad \ell_{12} = -\frac{(a - 1)(N - a)(\hat{\gamma}_1 - \hat{\gamma}_2)^2}{2 ((a - 1) \hat{\gamma}_1^2 + (N - a) \hat{\gamma}_2^2)} \]
One-way model
Likelihood ratio

- $\gamma_1 \leq \gamma_2 \Rightarrow$
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  \tilde{\gamma} = \frac{(a - 1) \hat{\gamma}_1^2 + (N - a) \hat{\gamma}_2^2}{(a - 1) \hat{\gamma}_1 + (N - a) \hat{\gamma}_2},
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- Under the null hypothesis
One-way model
Likelihood ratio

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  \ell_{12} = -\frac{(a - 1)(N - a)(\hat{\gamma}_1 - \hat{\gamma}_2)^2}{2 ((a - 1) \hat{\gamma}_1^2 + (N - a) \hat{\gamma}_2^2)}
  \]

- Under the null hypothesis
  \[
  \ell_0 = -\frac{a - 1}{2} \left( \frac{\hat{\gamma}_1}{\gamma} - 1 \right)^2 - \frac{N - a}{2} \left( \frac{\hat{\gamma}_2}{\gamma} - 1 \right)^2
  \]
One-way model
Likelihood ratio

- $\gamma_1 \leq \gamma_2 \Rightarrow$

$$\tilde{\gamma} = \frac{(a - 1) \hat{\gamma}_1^2 + (N - a) \hat{\gamma}_2^2}{(a - 1) \hat{\gamma}_1 + (N - a) \hat{\gamma}_2}, \quad \ell_{12} = -\frac{(a - 1)(N - a)(\hat{\gamma}_1 - \hat{\gamma}_2)^2}{2 ((a - 1) \hat{\gamma}_1^2 + (N - a) \hat{\gamma}_2^2)}$$

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$$\tilde{\gamma} = \frac{(a - 1) \hat{\gamma}_1^2 + (N - a) \hat{\gamma}_2^2}{(a - 1) \hat{\gamma}_1 + (N - a) \hat{\gamma}_2}, \quad \ell_0 = -\frac{(a - 1)(N - a)(\hat{\gamma}_1 - \hat{\gamma}_2)^2}{2 ((a - 1) \hat{\gamma}_1^2 + (N - a) \hat{\gamma}_2^2)}$$
One-way model
Likelihood ratio

Proposition

The LRT rejects the null hypothesis for large values of $\lambda$ given by:

$$\lambda = \begin{cases} 
0 
& \hat{\gamma}_1 \leq \hat{\gamma}_2 \\
\frac{(a-1)(N-a)(\hat{\gamma}_1-\hat{\gamma}_2)^2}{((a-1)\hat{\gamma}_1^2+(N-a)\hat{\gamma}_2^2)} 
& \hat{\gamma}_1 > \hat{\gamma}_2 
\end{cases}$$
One-way model
Random permutations

\[ y = [y_{ij}] - \text{lexicographical order} \]

- \( i \) – first factor
- \( j \) – replicates

Permutation

\[ y' = P_i y, \]

where \( P_i \) is a random permutation matrix.
One-way model
Permutation procedure

1. Calculate $\hat{\gamma}_1$ and $\hat{\gamma}_2$
2. Calculate $\lambda$
3. Generate a pseudo-sample of size $M$ of
   \[ \ell_1 = \ell(y_l) = \ell(P_l y) \]
4. Retrieve $c_\alpha$, the pseudo-sample’s empirical $(1 - \alpha)$-th quantile
5. Compare with the observed value $\lambda$ with $c_\alpha$
Combining the delta method (see van der Vaart, 1998) with the Self and Liang approach (Self and Liang, 1987) we obtain the following proposition.
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**Proposition**

The LRT rejects the null hypothesis $H_0$ for a large values of $\kappa$ given by

$$\kappa = \begin{cases} 0 & \hat{\gamma}_1 \leq \hat{\gamma}_2, \\ \frac{(a-1)(N-a)(\log \hat{\gamma}_1 - \log \hat{\gamma}_2)^2}{2(N-1)} & \hat{\gamma}_1 > \hat{\gamma}_2. \end{cases}$$
Combining the delta method (see van der Vaart, 1998) with the Self and Liang approach (Self and Liang, 1987) we obtain the following proposition.

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$$\kappa = \begin{cases} 0 & \hat{\gamma}_1 \leq \hat{\gamma}_2, \\ \frac{(a-1)(N-a)(\log \hat{\gamma}_1 - \log \hat{\gamma}_2)^2}{2(N-1)} & \hat{\gamma}_1 > \hat{\gamma}_2. \end{cases}$$

**Theorem**

The distribution of the LRT $\kappa$ is a mixture of $\chi^2_0$ and $\chi^2_1$ with coefficients $p$ and $1-p$, with

$$p = P(\hat{\gamma}_1 \leq \hat{\gamma}_2).$$
Two-fold nested model
Statistical model

\[ y = \mu 1_N + Z_1 \alpha + Z_2 \beta + e, \]

where

- \( y \) - vector of observed values,
- \( \mu \) - general mean,
- \( 1_N \) - vector of one’s,
- \( \alpha \) and \( \beta \) - vectors of random effects,
- \( Z_1 = I_a \otimes 1_b \otimes 1_r \) and \( Z_2 = I_a \otimes I_b \otimes 1_r \) - are design matrices,
- \( e \) - vector of errors

We assume that \( \alpha, \beta \) and \( e \) are indendent and that

\[
\alpha \sim N(0, \sigma_a^2 I_a), \quad \beta \sim N(0, \sigma_b^2 I_b) \quad \text{and} \quad e \sim N(0, \sigma_e^2 I_N),
\] respectively.
The problem of interest is:

\[ H_0 : \sigma_b^2 = 0 \quad \text{vs.} \quad H_1 : \sigma_b^2 > 0. \]
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\[ H_0 : \sigma_b^2 = 0 \quad \text{vs.} \quad H_1 : \sigma_b^2 > 0. \]

Alternatively, the problem of interest can be written as:

\[ H_0' : \gamma_2 = \gamma_3 \quad \text{vs.} \quad H_1' : \gamma_2 > \gamma_3, \]
The problem of interest is:

\[ H_0 : \sigma^2_b = 0 \quad \text{vs.} \quad H_1 : \sigma^2_b > 0. \]

Alternatively, the problem of interest can be written as:

\[ H'_0 : \gamma_2 = \gamma_3 \quad \text{vs.} \quad H'_1 : \gamma_2 > \gamma_3, \]

where \( \gamma_1 = br\sigma^2_a + r\sigma^2_b + \sigma^2_e \), \( \gamma_2 = r\sigma^2_b + \sigma^2_e \) and \( \gamma_3 = \sigma^2_e \), respectively (see Khuri et al, 1998).
The maximum likelihood estimators (MLE)

\[
\hat{\mu} = \frac{1}{N} 1'_N y, \quad \hat{\gamma}_1 = \frac{y' P_1 y}{a - 1}, \quad \hat{\gamma}_2 = \frac{y P_2 y}{a (b - 1)}, \quad \hat{\gamma}_3 = \frac{y' P_3 y}{ab (r - 1)},
\]

where

\[
P_1 = (I_a - \frac{1}{a} J_a) \otimes \frac{1}{br} J_{br}, \quad P_2 = I_a \otimes (I_b - \frac{1}{b} J_b) \otimes \frac{1}{r} J_r, \quad P_3 = I_{ab} \otimes (I_r - \frac{1}{r} J_r).
\]
Two-fold nested model

Statistical model

- The maximum likelihood estimators (MLE)

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\hat{\mu} = \frac{1}{N} 1_N' y, \quad \hat{\gamma}_1 = \frac{y' P_1 y}{a - 1}, \quad \hat{\gamma}_2 = \frac{y P_2 y}{a (b - 1)}, \quad \hat{\gamma}_3 = \frac{y' P_3 y}{ab (r - 1)},
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where \( P_1 = (I_a - \frac{1}{a} J_a) \otimes \frac{1}{br} J_{br}, P_2 = I_a \otimes (I_b - \frac{1}{b} J_b) \otimes \frac{1}{r} J_r, P_3 = I_{ab} \otimes (I_r - \frac{1}{r} J_r). \)

- \( E(\hat{\gamma}_1) = \gamma_1 \) and \( V(\hat{\gamma}_1) \approx \frac{2\gamma_1^2}{a}; \)
Two-fold nested model
Statistical model

- The maximum likelihood estimators (MLE)

\[ \hat{\mu} = \frac{1}{N} \mathbf{1}' \mathbf{y}, \quad \hat{\gamma}_1 = \frac{\mathbf{y}' \mathbf{P}_1 \mathbf{y}}{a - 1}, \quad \hat{\gamma}_2 = \frac{\mathbf{y} \mathbf{P}_2 \mathbf{y}}{a(b - 1)}, \quad \hat{\gamma}_3 = \frac{\mathbf{y}' \mathbf{P}_3 \mathbf{y}}{ab(r - 1)}, \]

where \( \mathbf{P}_1 = (\mathbf{I}_a - \frac{1}{a} \mathbf{J}_a) \otimes \frac{1}{br} \mathbf{J}_{br} \), \( \mathbf{P}_2 = \mathbf{I}_a \otimes (\mathbf{I}_b - \frac{1}{b} \mathbf{J}_b) \otimes \frac{1}{r} \mathbf{J}_r \), \( \mathbf{P}_3 = \mathbf{I}_{ab} \otimes (\mathbf{I}_r - \frac{1}{r} \mathbf{J}_r) \).

- \( \mathbb{E}(\hat{\gamma}_1) = \gamma_1 \) and \( \mathbb{V}(\hat{\gamma}_1) \approx \frac{2\gamma_1^2}{a} \); \( \mathbb{E}(\hat{\gamma}_2) = \gamma_2 \) and; \( \mathbb{V}(\hat{\gamma}_2) \approx \frac{2\gamma_2^2}{ab} \);
Two-fold nested model

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- The maximum likelihood estimators (MLE)

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\hat{\mu} = \frac{1}{N} 1_N' y, \quad \hat{\gamma}_1 = \frac{y' P_1 y}{a - 1}, \quad \hat{\gamma}_2 = \frac{y P_2 y}{a(b - 1)}, \quad \hat{\gamma}_3 = \frac{y' P_3 y}{ab(r - 1)},
\]

where \( P_1 = (I_a - \frac{1}{a} J_a) \otimes \frac{1}{br} J_{br}, \ P_2 = I_a \otimes \left( I_b - \frac{1}{b} J_b \right) \otimes \frac{1}{r} J_r, \)
\( P_3 = I_{ab} \otimes \left( I_r - \frac{1}{r} J_r \right). \)

- \( E(\hat{\gamma}_1) = \gamma_1 \) and \( V(\hat{\gamma}_1) \approx \frac{2\gamma_1^2}{a}; \ E(\hat{\gamma}_2) = \gamma_2 \) and; \( V(\hat{\gamma}_2) \approx \frac{2\gamma_2^2}{ab}; \)
- \( E(\hat{\gamma}_3) = \gamma_3 \) and; \( V(\hat{\gamma}_3) \approx \frac{2\gamma_3^2}{abr}. \)
Two-fold nested model
Statistical model

- The maximum likelihood estimators (MLE)

\[
\hat{\mu} = \frac{1}{N} \mathbf{1}'_N \mathbf{y}, \quad \hat{\gamma}_1 = \frac{\mathbf{y}' \mathbf{P}_1 \mathbf{y}}{a - 1}, \quad \hat{\gamma}_2 = \frac{\mathbf{y} \mathbf{P}_2 \mathbf{y}}{a (b - 1)}, \quad \hat{\gamma}_3 = \frac{\mathbf{y}' \mathbf{P}_3 \mathbf{y}}{ab (r - 1)},
\]

where \( \mathbf{P}_1 = (\mathbf{I}_a - \frac{1}{a} \mathbf{J}_a) \otimes \frac{1}{br} \mathbf{J}_{br} \), \( \mathbf{P}_2 = \mathbf{I}_a \otimes (\mathbf{I}_b - \frac{1}{b} \mathbf{J}_b) \otimes \frac{1}{r} \mathbf{J}_r \), \( \mathbf{P}_3 = \mathbf{I}_{ab} \otimes (\mathbf{I}_r - \frac{1}{r} \mathbf{J}_r) \).

- \( E(\hat{\gamma}_1) = \gamma_1 \) and \( V(\hat{\gamma}_1) \approx \frac{2\gamma_1^2}{a} \); \( E(\hat{\gamma}_2) = \gamma_2 \) and; \( V(\hat{\gamma}_2) \approx \frac{2\gamma_2^2}{ab} \); \( E(\hat{\gamma}_3) = \gamma_3 \) and; \( V(\hat{\gamma}_3) \approx \frac{2\gamma_3^2}{abr} \).

- \( \mathbf{V}\hat{\boldsymbol{\gamma}} = \text{diag} \left( \frac{2\gamma_1^2}{a}, \frac{2\gamma_2^2}{ab}, \frac{2\gamma_3^2}{abr} \right) \)
The maximum likelihood estimators (MLE)

\[
\hat{\mu} = \frac{1}{N} 1'_N y, \quad \hat{\gamma}_1 = \frac{y' P_1 y}{a - 1}, \quad \hat{\gamma}_2 = \frac{y P_2 y}{a (b - 1)}, \quad \hat{\gamma}_3 = \frac{y' P_3 y}{ab (r - 1)},
\]

where \( P_1 = (I_a - \frac{1}{a} J_a) \otimes \frac{1}{br} J_{br}, \ P_2 = I_a \otimes (I_b - \frac{1}{b} J_b) \otimes \frac{1}{r} J_r, \ P_3 = I_{ab} \otimes (I_r - \frac{1}{r} J_r). \)

- \( E (\hat{\gamma}_1) = \gamma_1 \) and \( V (\hat{\gamma}_1) \approx \frac{2\gamma_1^2}{a}; \ E (\hat{\gamma}_2) = \gamma_2 \) and; \( V (\hat{\gamma}_2) \approx \frac{2\gamma_2^2}{ab}; \)
- \( E (\hat{\gamma}_3) = \gamma_3 \) and; \( V (\hat{\gamma}_3) \approx \frac{2\gamma_3^2}{abr}. \)
- \( V \hat{\gamma} = diag \left( \frac{2\gamma_1^2}{a}, \frac{2\gamma_2^2}{ab}, \frac{2\gamma_3^2}{abr} \right) \)
- \( \mathbf{V}^{-1/2} (\hat{\gamma} - \gamma) \xrightarrow{d} N (0, I_3). \)
Two-fold nested model
Likelihood ratio. $H_1$

\[ \gamma_1 \geq \gamma_2 \geq \gamma_3 \]

\[ \tilde{\gamma}_1 = \hat{\gamma}_1 \]
\[ \tilde{\gamma}_2 = \hat{\gamma}_2 \]
\[ \tilde{\gamma}_3 = \hat{\gamma}_3 \]

\[ -2\ell_1 = 0 \]
Likelihood Ratio

\( H_1 \)

\( \hat{\gamma}_1 < \hat{\gamma}_2, \hat{\gamma}_2 \geq \hat{\gamma}_3 \)

\[
\tilde{\gamma}_{12} = \frac{\hat{\gamma}_1^2 + b\hat{\gamma}_2^2}{\hat{\gamma}_1 + b\hat{\gamma}_2} \\
\tilde{\gamma}_3 = \hat{\gamma}_3
\]

\[-2\ell_1 = \frac{ab^2(\hat{\gamma}_1\hat{\gamma}_2 - \hat{\gamma}_2^2)^2 + ab(\hat{\gamma}_1\hat{\gamma}_2 - \hat{\gamma}_1^2)^2}{(\hat{\gamma}_1^2 + b\hat{\gamma}_2^2)^2} \]
Two-fold nested model
Likelihood ratio. $H_1$

\[ \hat{\gamma}_1 \geq \frac{\hat{\gamma}_2^2 + r\hat{\gamma}_3^2}{\hat{\gamma}_2 + r\hat{\gamma}_3}, \hat{\gamma}_2 < \hat{\gamma}_3 \]

\[ \tilde{\gamma}_1 = \hat{\gamma}_1 \]
\[ \tilde{\gamma}_{23} = \frac{\hat{\gamma}_2^2 + r\hat{\gamma}_3^2}{\hat{\gamma}_2 + r\hat{\gamma}_3} \]

\[ -2 \ell_1 = \frac{abr^2(\hat{\gamma}_2\hat{\gamma}_3 - \hat{\gamma}_3^2)^2 + abr(\hat{\gamma}_2\hat{\gamma}_3 - \hat{\gamma}_3^2)^2}{(\hat{\gamma}_2 + r\hat{\gamma}_3^2)^2} \]
Two-fold nested model
Likelihood ratio. $H_1$

\[
\hat{\gamma}_1 < \frac{\hat{\gamma}_2 + r\hat{\gamma}_3}{\hat{\gamma}_2 + r\hat{\gamma}_3}, \quad \hat{\gamma}_2 < \hat{\gamma}_3
\]

\[
\hat{\gamma}_{123} = \frac{\hat{\gamma}_1^2 + b\hat{\gamma}_2^2 + br\hat{\gamma}_3^2}{\hat{\gamma}_1 + b\hat{\gamma}_2 + br\hat{\gamma}_3}
\]

\[
-2\ell_1 = \frac{a(b(\hat{\gamma}_1\hat{\gamma}_2 - \hat{\gamma}_2^2) + br(\hat{\gamma}_1\hat{\gamma}_3 - \hat{\gamma}_3^2))^2}{(\hat{\gamma}_1^2 + b\hat{\gamma}_2^2 + br\hat{\gamma}_3^2)^2}
\]

\[
+ \frac{ab((\hat{\gamma}_1\hat{\gamma}_2 - \hat{\gamma}_1^2) + br(\hat{\gamma}_2\hat{\gamma}_3 - \hat{\gamma}_3^2))^2}{(\hat{\gamma}_1^2 + b\hat{\gamma}_2^2 + br\hat{\gamma}_3^2)^2}
\]

\[
+ \frac{abr((\hat{\gamma}_1\hat{\gamma}_2 - \hat{\gamma}_1^2) + b(\hat{\gamma}_1\hat{\gamma}_2 - \hat{\gamma}_2^2))^2}{(\hat{\gamma}_1^2 + b\hat{\gamma}_2^2 + br\hat{\gamma}_3^2)^2}
\]
Two-fold nested model
Likelihood ratio. $H_0 : \gamma_2 = \gamma_3$

\[
\hat{\gamma}_1 \geq \frac{\hat{\gamma}_2^2 + r\hat{\gamma}_3^2}{\hat{\gamma}_2 + r\hat{\gamma}_3}
\]

\[
\tilde{\gamma}_1 = \hat{\gamma}_1
\]

\[
\tilde{\gamma}_{23} = \frac{\hat{\gamma}_2^2 + r\hat{\gamma}_3^2}{\hat{\gamma}_2 + r\hat{\gamma}_3}
\]

\[
-2\ell_1 = abr^2(\hat{\gamma}_2 \hat{\gamma}_3 - \hat{\gamma}_3^2)^2 + abr(\hat{\gamma}_2 \hat{\gamma}_3 - \hat{\gamma}_2^2)^2
\]

\[
(\hat{\gamma}_2^2 + r\hat{\gamma}_3^2)^2
\]
Two-fold nested model
Likelihood ratio. $H_0: \hat{\gamma}_2 = \hat{\gamma}_3$

\[
\hat{\gamma}_1 < \frac{\hat{\gamma}_2 + r \hat{\gamma}_3}{\hat{\gamma}_2 + r \hat{\gamma}_3}
\]

\[
\tilde{\gamma}_{123} = \frac{\hat{\gamma}_1 + b \hat{\gamma}_2 + br \hat{\gamma}_3}{\hat{\gamma}_1 + b \hat{\gamma}_2 + br \hat{\gamma}_3}
\]

\[
-2\ell_0 = a \left( b (\hat{\gamma}_1 \hat{\gamma}_2 - \hat{\gamma}_2^2) + br(\hat{\gamma}_1 \hat{\gamma}_3 - \hat{\gamma}_3^2) \right)^2 / \left( \hat{\gamma}_1^2 + b \hat{\gamma}_2^2 + br \hat{\gamma}_3^2 \right)^2
\]

\[
+ ab \left( (\hat{\gamma}_1 \hat{\gamma}_2 - \hat{\gamma}_1^2) + br(\hat{\gamma}_2 \hat{\gamma}_3 - \hat{\gamma}_3^2) \right)^2 / \left( \hat{\gamma}_1^2 + b \hat{\gamma}_2^2 + br \hat{\gamma}_3^2 \right)^2
\]

\[
+ abr \left( (\hat{\gamma}_1 \hat{\gamma}_2 - \hat{\gamma}_1^2) + b(\hat{\gamma}_1 \hat{\gamma}_2 - \hat{\gamma}_2^2) \right)^2 / \left( \hat{\gamma}_1^2 + b \hat{\gamma}_2^2 + br \hat{\gamma}_3^2 \right)^2
\]
Two-fold nested model
Random Permutations

\[ y = [y_{ijk}] \text{ – lexicographical order} \]

- \( i \) – first factor
- \( j \) – second factor
- \( k \) – replicates

Selective permutation

\[ y_l = (I_a \otimes P_l)y, \]

where \( P_l \) is a random permutation matrix.
Two-fold nested model
Permutation Procedure

1. Calculate $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\gamma}_3$
2. Calculate $\lambda$
3. Generate a pseudo-sample of size $M$ of
   \[ \ell_l = \ell(y_l) = \ell((I_a \otimes P_l)y) \]
4. Retrieve $c_\alpha$, the pseudo-sample’s empirical $(1 - \alpha)$-th quantile
5. Compare the observed value of $\lambda$ with $c_\alpha$
Simulation studies
One-way model

(i) $N(0, 1)$,
Simulation studies
One-way model

- $(i) \ N(0, 1)$, $(ii) \ G(10, 5)$,
Simulation studies
One-way model

• (i) $N(0, 1)$, (ii) $G(10, 5)$, (iii) $t - Stud.(5)$
Simulation studies
One-way model

- (i) \( N(0, 1) \), (ii) \( G(10, 5) \), (iii) \( t - Stud.(5) \)
- \( a = 20, \ b = 10 \)
Simulation studies
One-way model

- (i) $N(0,1)$, (ii) $G(10,5)$, (iii) $t - \text{Stud.}(5)$
- $a = 20$, $b = 10$
- $\mu = 1$, $\sigma^2_e = 1$
Simulation studies
One-way model

- (i) $N(0, 1)$
- (ii) $G(10, 5)$
- (iii) $t - Stud.(5)$

- $a = 20$, $b = 10$
- $\mu = 1$, $\sigma^2_e = 1$
- 1000 simulations runs

as a reference we took F-test
Simulation studies
One-way model

- (i) $N(0, 1)$, (ii) $G(10, 5)$, (iii) $t - Stud.(5)$
- $a = 20$, $b = 10$
- $\mu = 1$, $\sigma^2_e = 1$
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- 1000 permutations
Simulation studies
One-way model

- $(i) \, N(0, 1)$, $(ii) \, G(10, 5)$, $(iii) \, t - Stud.(5)$
- $a = 20$, $b = 10$
- $\mu = 1$, $\sigma_e^2 = 1$
- 1000 simulations runs
- 1000 permutations
- as a reference we took F-test
## Simulation studies

### One-way model

<table>
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<th>$G(10, 5)$, $G(10, 5)$</th>
<th>$t - St.(5)$, $t - St.(5)$</th>
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Simulation studies
Two-fold nested model

- (i) $N(0, 1)$, (ii) $G(10, 5)$, (iii) $t - \text{Stud.}(5)$
(i) $N(0, 1)$, (ii) $G(10, 5)$, (iii) $t - Stud.(5)$

$a = 20$, $b = 5$, $r = 5$
Similation studies
Two-fold nested model

- (i) $N(0, 1)$, (ii) $G(10, 5)$, (iii) $t - Stud.(5)$
- $a = 20, b = 5, r = 5$
- $\mu = 1, \sigma_a^2 = 1, \sigma_e^2 = 1$
Simulation studies
Two-fold nested model

- (i) $N(0, 1)$,
- (ii) $G(10, 5)$,
- (iii) $t - \text{Stud.}(5)$

- $a = 20$, $b = 5$, $r = 5$
- $\mu = 1$, $\sigma^2_a = 1$, $\sigma^2_e = 1$
- 250 simulations runs
Simulation studies

Two-fold nested model

- (i) $N(0, 1)$, (ii) $G(10, 5)$, (iii) $t - Stud.(5)$
- $a = 20, \ b = 5, \ r = 5$
- $\mu = 1, \ \sigma_a^2 = 1, \ \sigma_e^2 = 1$
- 250 simulations runs
- 1000 permutations
Similation studies
Two-fold nested model

- (i) $N(0, 1)$, (ii) $G(10, 5)$, (iii) $t - Stud.(5)$
- $a = 20$, $b = 5$, $r = 5$
- $\mu = 1$, $\sigma^2_a = 1$, $\sigma^2_e = 1$
- 250 simulations runs
- 1000 permutations
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Simulation studies
Two-fold nested model

(i) $N(0, 1)$
Simulation studies
Two-fold nested model

\[(ii) \ G(10, 1)\]
Simulation studies
Two-fold nested model

(iii) $t - \text{Stud.}(5)$
We developed a permutation test procedure for one-way and two-fold nested model.
Summary

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The proposed permutation test procedure performed similar as the F-test for the one-way model. For the two-fold nested model it somewhat surpasses the F-test, up until some break point.
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Outlook
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Outlook
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Outlook
- Perform simulation studies for increased numbers $b$ and $r$.
- Perform simulations for $t - Stud$ with increased degrees of freedom.


