Logarithmic interpolation spaces and Besov spaces

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PDEs, Potential Theory and Function Spaces
In honour of Lars Inge Hedberg
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If $\Omega = \mathbb{Z}$, we obtain the Lorentz-Zygmund sequence spaces $\ell_{p,q}(\log \ell)_\gamma$.

Logarithmic interpolation methods:

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$J(t,a) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\} \quad , \quad a \in A_0 \cap A_1$.

▷ F. Cobos and A. Segurado, J. Funct. Anal. 268 (2015) 2906-2945.

Density property.

- $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,q}$ if $q < \infty$.

- Assume α_∞ satisfies the same condition as α_0

$$(1) \qquad \begin{cases} \alpha_\infty + 1/q < 0 & \text{if} \quad q < \infty, \\ \alpha_\infty \leq 0 & \text{if} \quad q = \infty. \end{cases}$$

Then $(\ell_1, \ell_\infty)_{1,q,\mathbb{A}} = \ell_\infty$.

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⋆ There is no J-representation in the range (1).

⋆ If $\alpha_\infty + 1/q > 0$ and $1 \leq q \leq \infty$, then $(A_0, A_1)_{1,q,(\alpha_0,\alpha_\infty)}$ consists of all those $a \in A_0 + A_1$ for which there is a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty u(t)\frac{dt}{t} \quad \text{(convergence in } A_0 + A_1)$$

with

$$\Big(\int_0^\infty [t^{-1}J(t,u(t))\ell^{(1+\alpha_0,1+\alpha_\infty)}(t)]^q \frac{dt}{t}\Big)^{1/q} < \infty.$$

Moreover,

$$\|a\| = \inf\Big\{\Big(\int_0^\infty [t^{-1}J(t,u(t))\ell^{(1+\alpha_0,1+\alpha_\infty)}(t)]^q \frac{dt}{t}\Big)^{1/q} : a = \int_0^\infty u(t)\frac{dt}{t}\Big\}$$

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⋆ If $1 \le q < \infty$ and $\alpha_\infty = -1/q$, in addition to the correction of the exponents, one should insert an iterated logarithm in the part of the integral on $(1, \infty)$.

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- If $0 < \theta < 1$ and $(\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, then $(A_0, A_1)'_{\theta,q,(\alpha_0,\alpha_\infty)} = (A_0', A_1')_{\theta,q',(-\alpha_\infty,-\alpha_0)}$

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Theorem.-Let $(\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ such that $\alpha_0 + 1/q < 0$.

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(b) If $\alpha_\infty = -1/q$ then $(A_0, A_1)'_{1,q,(\alpha_0,-1/q)}$

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- J-representations and duality for $(A_0, A_1)_{0,q,(\alpha_0,\alpha_\infty)}$.

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This method is also called the limiting real method and it is denoted by

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▷ F. Cobos, L.M. Fernández-Cabrera, T. Kühn and T. Ullrich, J. Funct. Anal. 256 (2009) 2321-2366.

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▷ R.A. DeVore, S.D. Riemenschneider and R.C. Sharpley, J. Funct. Anal. 33 (1979) 58-94.

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▷ W. Farkas and H.-G. Leopold, Ann. Mat. Pura Appl. 185 (2006) 1-62.

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The k-th order modulus of smoothness of $f \in L_p = L_p(\mathbb{R}^n)$ is

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Here $k \in \mathbb{N}, h \in \mathbb{R}^n$,

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- If $s > 0$ then $\mathbb{B}_{p,q}^{s,b} = B_{p,q}^{s,b}$

▷ H. Triebel, Birkhäuser, 1983 (case $b = 0$).

▷ D.D. Haroske and S. Moura, J. Approx. Theory 128 (2004) 151-174 (general b).

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Theorem.- Let $1 < p < \infty, 1 \leq q < \infty$ and $b > -1/q$. The space $(\mathbb{B}^{0,b}_{p,q})'$ consists of all $f \in H^{-1}_{p'}$ such that $I_{-1} f \in \mathrm{Lip}^{(1,-b-1)}_{p',q'}$ with $1/p + 1/p' = 1 = 1/q + 1/q'$. Moreover $\|f\|_{(\mathbb{B}^{0,b}_{p,q})'} \sim \|I_{-1} f\|_{\mathrm{Lip}^{(1,-b-1)}_{p',q'}}$.

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Fourier coefficients

$$\hat{f}(m) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-imx}dx \quad , \quad m \in \mathbb{Z}.$$

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$$\mathfrak{F} : \mathbb{B}^{0,b}_{p,q}(\mathbb{T}) = (L_p(\mathbb{T}), \mathbb{B}^{\alpha}_{p,p}(\mathbb{T}))_{(0,b),q} \longrightarrow (\ell_{p'}, \ell_{r,p})_{(0,b),q} \text{ boundedly.}$$

Furthermore, for $\theta/r = 1/p'$
$(\ell_{p'}, \ell_{r,p})_{(0,b),q} = ((\ell_\infty, \ell_{r,p})_{\theta,p'}, \ell_{r,p})_{(0,b),q}$

$$\boxed{((A_0, A_1)_{\theta,r}, A_1)_{(0,\eta),q} \hookrightarrow (A_0, A_1)_{\theta,q,\eta+1/\max\{r,q\}}} \quad \hookrightarrow (\ell_\infty, \ell_{r,p})_{\theta,q,b+1/\max\{p',q\}}$$

$$= \ell_{p',q}(\log \ell)_{b+1/\max\{p',q\}}.$$

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