# Reiteration of approximation spaces

Óscar Domínguez

joint work with Fernando Cobos

Universidad Complutense de Madrid

June, 2015

# Approximation spaces

Let $X$ be a quasi-Banach space. An approximation family in $X$ is a sequence $(G_n)_{n\in\mathbb{N}_0}$ formed by subsets of $X$ such that the following conditions hold

$$G_0 = 0 \text{ and } G_n \subseteq G_{n+1} \text{ for } n \in \mathbb{N}_0,$$

$$\lambda G_n \subseteq G_n \text{ for any scalar } \lambda \text{ and } n \in \mathbb{N},$$

$$G_n + G_m \subseteq G_{n+m} \text{ for any } n, m \in \mathbb{N}.$$

# Approximation spaces

Let $X$ be a quasi-Banach space. An approximation family in $X$ is a sequence $(G_n)_{n\in\mathbb{N}_0}$ formed by subsets of $X$ such that the following conditions hold

$$G_0 = 0 \text{ and } G_n \subseteq G_{n+1} \text{ for } n \in \mathbb{N}_0,$$

$$\lambda G_n \subseteq G_n \text{ for any scalar } \lambda \text{ and } n \in \mathbb{N},$$

$$G_n + G_m \subseteq G_{n+m} \text{ for any } n, m \in \mathbb{N}.$$

Given any $f \in X$ and $n \in \mathbb{N}$, the $n$-th approximation error of $f$ is given by

$$E_n(f) = E_n(f; X) = \inf\{\|f - g\|_X : g \in G_{n-1}\}.$$

Let $\alpha > 0$ and $0 < p \leqslant \infty$. The approximation space $X_p^\alpha = (X, G_n)_p^\alpha$ is the set of all $f \in X$ which have a finite quasi-norm

$$\|f\|_{X_p^\alpha} = \left( \sum_{n=1}^{\infty} (n^\alpha E_n(f))^p n^{-1} \right)^{1/p}.$$


▻ A. Pietsch, J. Approx. Theory 32 (1981) 115–134.

▻ P.L. Butzer, K. Scherer, Mannheim, 1968.

▻ Yu.A. Brudnyĭ, Jaroslavl, 1977.

▻ Yu.A. Brudnyĭ, N. Krugljak, Jaroslavl, 1978.

▻ R.A. DeVore, G.G. Lorentz, Springer, Berlin, 1993.

# Examples

• Let $X = \ell_\infty$ and $G_n = s_n$, the subset of sequences having at most $n$ coordinates different from 0, then

$$E_n(\xi) = |\xi_n^*| \text{ and } (\ell_\infty)_p^\alpha = \ell_{1/\alpha,p}.$$

# Examples

• Let $X = \ell_\infty$ and $G_n = s_n$, the subset of sequences having at most $n$ coordinates different from 0, then

$$E_n(\xi) = |\xi_n^*| \text{ and } (\ell_\infty)_p^\alpha = \ell_{1/\alpha,p}.$$

• Let $E$ and $F$ be Banach spaces. If $X = \mathfrak{L}(E,F)$ and $G_n = \mathfrak{F}_n(E,F)$, then

$$E_n(T) = a_n(T) \text{ and } (\mathfrak{L}(E,F))_p^\alpha = \mathfrak{L}_{1/\alpha,p}(E,F).$$

# Examples

• Let $X = \ell_\infty$ and $G_n = s_n$, the subset of sequences having at most $n$ coordinates different from 0, then

$$E_n(\xi) = |\xi_n^*| \text{ and } (\ell_\infty)_p^\alpha = \ell_{1/\alpha,p}.$$

• Let $E$ and $F$ be Banach spaces. If $X = \mathfrak{L}(E,F)$ and $G_n = \mathfrak{F}_n(E,F)$, then

$$E_n(T) = a_n(T) \text{ and } (\mathfrak{L}(E,F))_p^\alpha = \mathfrak{L}_{1/\alpha,p}(E,F).$$

• Let $0 < p \leqslant \infty$. If $X = L^p(\mathbb{T})$ and $G_n = T_n$, the subset of all trigonometric polynomials with degree less than or equal to $n$, that is,

$$T_n = \left\{ \sum_{k=-n}^{n} c_k e^{ikx} : c_k \in \mathbb{C} \right\}.$$

Then, $(L^p(\mathbb{T}))_q^\alpha = \mathbf{B}_{p,q}^\alpha(\mathbb{T})$.

▻ H.-J. Schmeisser, H. Triebel, Wiley, Chichester, 1987.

Let $\alpha > 0$ and $0 < p \leqslant \infty$. The approximation space $X_p^\alpha = (X, G_n)_p^\alpha$ is the set of all $f \in X$ which have a finite quasi-norm

$$\|f\|_{X_p^\alpha} = \left( \sum_{n=1}^{\infty} (n^\alpha E_n(f))^p n^{-1} \right)^{1/p}.$$

Let $-\infty < \gamma < \infty$ and $0 < p \leqslant \infty$. The limiting approximation space $X_p^{(0,\gamma)} = (X, G_n)_p^{(0,\gamma)}$ is formed by all elements $f \in X$ such that

$$\|f\|_{X_p^{(0,\gamma)}} = \left( \sum_{n=1}^{\infty} ((1 + \log n)^{\gamma} E_n(f))^p n^{-1} \right)^{1/p} < \infty.$$

Let $-\infty < \gamma < \infty$ and $0 < p \leqslant \infty$. The limiting approximation space $X_p^{(0,\gamma)} = (X, G_n)_p^{(0,\gamma)}$ is formed by all elements $f \in X$ such that

$$\|f\|_{X_p^{(0,\gamma)}} = \left( \sum_{n=1}^{\infty} ((1 + \log n)^{\gamma} E_n(f))^p n^{-1} \right)^{1/p} < \infty.$$


▻ F. Cobos, I. Resina, J. London Math. Soc. 39 (1989) 324–334.

▻ F. Cobos, M. Milman, Numer. Funct. Anal. Optim. 11 (1990) 11–31.

▻ F. Fehér, G. Grässler, J. Comput. Anal. Appl. 3 (2001) 95–108.

Let $-\infty < \gamma < \infty$ and $0 < p \leqslant \infty$. The limiting approximation space $X_p^{(0,\gamma)} = (X, G_n)_p^{(0,\gamma)}$ is formed by all elements $f \in X$ such that

$$\|f\|_{X_p^{(0,\gamma)}} = \left( \sum_{n=1}^{\infty} ((1 + \log n)^{\gamma} E_n(f))^p n^{-1} \right)^{1/p} < \infty.$$

⊳ F. Cobos, I. Resina, J. London Math. Soc. 39 (1989) 324–334.

⊳ F. Cobos, M. Milman, Numer. Funct. Anal. Optim. 11 (1990) 11–31.

⊳ F. Fehér, G. Grässler, J. Comput. Anal. Appl. 3 (2001) 95–108.

Note that $X_p^{(0,\gamma)} = X$ if $\gamma < -1/p$. Moreover, the following continuous embeddings hold

$$X_p^{\alpha} \hookrightarrow X_q^{(0,\gamma)} \hookrightarrow X$$

for any choice of parameters.

# Examples

- Let $X = \ell_\infty$ and $G_n = s_n$. Then, $(\ell_\infty)_p^{(0,\gamma)} = \ell_{\infty,p}(\log \ell)_\gamma$.

# Examples

- Let $X = \ell_\infty$ and $G_n = s_n$. Then, $(\ell_\infty)_p^{(0,\gamma)} = \ell_{\infty,p}(\log \ell)_\gamma$.
- If $X = \mathfrak{L}(E,F)$ and $G_n = \mathfrak{F}_n(E,F)$. Then, $(\mathfrak{L}(E,F))_p^{(0,\gamma)} = \mathfrak{L}_{\infty,p,\gamma}(E,F)$.

# Examples

• Let $X = \ell_\infty$ and $G_n = s_n$. Then, $(\ell_\infty)_p^{(0,\gamma)} = \ell_{\infty,p}(\log \ell)_\gamma$.

• If $X = \mathfrak{L}(E,F)$ and $G_n = \mathfrak{F}_n(E,F)$. Then, $(\mathfrak{L}(E,F))_p^{(0,\gamma)} = \mathfrak{L}_{\infty,p,\gamma}(E,F)$.

• Let $0 < p \leqslant \infty$. If $X = L^p(\mathbb{T})$ and $G_n = T_n$. Then, $(L^p(\mathbb{T}))_q^{(0,\gamma)} = \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$ where

$$\|f\|_{\mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})} + \left(\int_0^1 ((1-\log t)^\gamma \omega(f,t)_p)^q \frac{dt}{t}\right)^{1/q}.$$


▻ R.A. DeVore, S.D. Riemenschneider, R.C. Sharpley, J. Funct. Anal. 33 (1979) 58–94.

▻ F. Cobos, O. Domínguez, Studia Math. 223 (2014) 193–204.

The theory of limiting approximation spaces does not follow by taking $\alpha = 0$ in the classical theory.

The theory of limiting approximation spaces does not follow by taking $\alpha = 0$ in the classical theory.

• Representation theorem (Pietsch).- An element $f \in X$ belongs to $X_p^\alpha$ if and only if

$$f = \sum_{k=0}^{\infty} g_k, g_k \in G_{2^k}, \tag{1}$$

with

$$\sum_{k=0}^{\infty} (2^{k\alpha} \|g_k\|_X)^p < \infty. \tag{2}$$

Moreover,

$$\|f\|_{X_p^\alpha} \sim \inf \left( \sum_{k=0}^{\infty} (2^{k\alpha} \|g_k\|_X)^p \right)^{1/p},$$

where the infimum is taking over all possible representations (1) such that (2) holds.

The theory of limiting approximation spaces does not follow by taking $\alpha = 0$ in the classical theory.

$$f = \sum_{k=0}^{\infty} g_k, g_k \in G_{2^k}, \|f\|_{X_p^\alpha} \sim \inf \left( \sum_{k=0}^{\infty} (2^{k\alpha} \|g_k\|_X)^p \right)^{1/p}.$$

The theory of limiting approximation spaces does not follow by taking $\alpha = 0$ in the classical theory.

$$f = \sum_{k=0}^{\infty} g_k, g_k \in G_{2^k}, \|f\|_{X_p^\alpha} \sim \inf \left( \sum_{k=0}^{\infty} (2^{k\alpha} \|g_k\|_X)^p \right)^{1/p}.$$

• Limiting representation theorem (Cobos-Resina, Fehér-Grässler).- Let $\gamma > -1/p$ and $\mu_k = 2^{2^k}, k = 0, 1, \dots$. An element $f \in X$ belongs to $X_p^{(0,\gamma)}$ if and only if

$$f = \sum_{k=0}^{\infty} g_k, g_k \in G_{\mu_k}, \tag{3}$$

with

$$\sum_{k=0}^{\infty} (2^{k(\gamma+1/p)} \|g_k\|_X)^p < \infty. \tag{4}$$

Moreover,

$$\|f\|_{X_p^{(0,\gamma)}} \sim \inf \left( \sum_{k=0}^{\infty} (2^{k(\gamma+1/p)} \|g_k\|_X)^p \right)^{1/p},$$

where the infimum is taking over all possible representations (3) such that (4) holds.

# Reiteration of approximation constructions

Let $(G_n)$ be an approximation scheme in $X$. Since $G_n \subseteq X_p^{\alpha}$ and $G_n \subseteq X_q^{(0,\gamma)}$ for any $n \in \mathbb{N}_0$. Then, $(G_n)$ determines an approximation scheme in $X_p^{\alpha}$ and $X_q^{(0,\gamma)}$.

# Reiteration of approximation constructions

Let $(G_n)$ be an approximation scheme in $X$. Since $G_n \subseteq X_p^\alpha$ and $G_n \subseteq X_q^{(0,\gamma)}$ for any $n \in \mathbb{N}_0$. Then, $(G_n)$ determines an approximation scheme in $X_p^\alpha$ and $X_q^{(0,\gamma)}$.

Reiteration theorem (Pietsch).- Let $\alpha, \beta > 0$ and $0 < p, r \leqslant \infty$. Then, we have with equivalence of quasi-norms

$$(X_p^\alpha)_r^\beta = X_r^{\alpha+\beta}.$$

# Reiteration of approximation constructions

Let $(G_n)$ be an approximation scheme in $X$. Since $G_n \subseteq X_p^\alpha$ and $G_n \subseteq X_q^{(0,\gamma)}$ for any $n \in \mathbb{N}_0$. Then, $(G_n)$ determines an approximation scheme in $X_p^\alpha$ and $X_q^{(0,\gamma)}$.

Reiteration theorem (Pietsch).- Let $\alpha, \beta > 0$ and $0 < p, r \leqslant \infty$. Then, we have with equivalence of quasi-norms

$$(X_p^\alpha)_r^\beta = X_r^{\alpha+\beta}.$$

Reiteration theorem (Fehér-Grässler).- Let $0 < q, r \leqslant \infty, \gamma > -1/q$ and $\delta > -1/r$. Then, we have with equivalence of quasi-norms

$$(X_q^{(0,\gamma)})_r^{(0,\delta)} = X_r^{(0,\gamma+1/q+\delta)}.$$

We study the stability properties when we apply first the construction $(\cdot)_p^\alpha$ and then $(\cdot)_q^{(0,\gamma)}$ or vice versa.

▻ F. Cobos, O. Domínguez, J. Approx. Theory 189 (2015) 43–66.

We study the stability properties when we apply first the construction $(\cdot)_p^{\alpha}$ and then $(\cdot)_q^{(0,\gamma)}$ or vice versa.

$\rhd$ F. Cobos, O. Domínguez, J. Approx. Theory 189 (2015) 43–66.

THEOREM.- Suppose that $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma > -1/q$. Then, $(X_q^{(0,\gamma)})_p^{\alpha} = X_p^{(\alpha,\gamma+1/q)}$ with equivalence of quasi-norms, where

$$\|f\|_{X_p^{(\alpha,\gamma+1/q)}} = \left( \sum_{n=1}^{\infty} (n^{\alpha}(1+\log n)^{\gamma+1/q} E_n(f;X))^p n^{-1} \right)^{1/p}.$$

We study the stability properties when we apply first the construction $(\cdot)_p^\alpha$ and then $(\cdot)_q^{(0,\gamma)}$ or vice versa.

⊳ F. Cobos, O. Domínguez, J. Approx. Theory 189 (2015) 43–66.

THEOREM.- Suppose that $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma > -1/q$. Then, $(X_q^{(0,\gamma)})_p^\alpha = X_p^{(\alpha,\gamma+1/q)}$ with equivalence of quasi-norms, where

$$\|f\|_{X_p^{(\alpha,\gamma+1/q)}} = \left(\sum_{n=1}^{\infty} (n^\alpha(1+\log n)^{\gamma+1/q} E_n(f;X))^p n^{-1}\right)^{1/p}.$$

⊳ E. Pustylnik, Collect. Math. 57 (2006) 257–277.

We study the stability properties when we apply first the construction $(\cdot)_p^\alpha$ and then $(\cdot)_q^{(0,\gamma)}$ or vice versa.

▻ F. Cobos, O. Domínguez, J. Approx. Theory 189 (2015) 43–66.

THEOREM.- Suppose that $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma > -1/q$. Then, $(X_q^{(0,\gamma)})_p^\alpha = X_p^{(\alpha,\gamma+1/q)}$ with equivalence of quasi-norms, where

$$\|f\|_{X_p^{(\alpha,\gamma+1/q)}} = \left(\sum_{n=1}^{\infty} (n^\alpha (1+\log n)^{\gamma+1/q} E_n(f;X))^p n^{-1}\right)^{1/p}.$$

PROOF (OUTLINE).- The embedding $(X_q^{(0,\gamma)})_p^\alpha \hookrightarrow X_p^{(\alpha,\gamma+1/q)}$ is obtained via Jackson-type inequality

$$E_{2n-1}(f;X) \lesssim (1+\log n)^{-(\gamma+1/q)} E_n(f;X_q^{(0,\gamma)}), f \in X_q^{(0,\gamma)}, n \in \mathbb{N}.$$

We study the stability properties when we apply first the construction $(\cdot)_p^{\alpha}$ and then $(\cdot)_q^{(0,\gamma)}$ or vice versa.

$\rhd$ F. Cobos, O. Domínguez, J. Approx. Theory 189 (2015) 43–66.

THEOREM.- Suppose that $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma > -1/q$. Then, $(X_q^{(0,\gamma)})_p^{\alpha} = X_p^{(\alpha,\gamma+1/q)}$ with equivalence of quasi-norms, where

$$\|f\|_{X_p^{(\alpha,\gamma+1/q)}} = \left(\sum_{n=1}^{\infty} (n^{\alpha}(1+\log n)^{\gamma+1/q} E_n(f;X))^p n^{-1}\right)^{1/p}.$$

PROOF (OUTLINE).- The embedding $(X_q^{(0,\gamma)})_p^{\alpha} \hookrightarrow X_p^{(\alpha,\gamma+1/q)}$ is obtained via Jackson-type inequality

$$E_{2n-1}(f;X) \lesssim (1+\log n)^{-(\gamma+1/q)} E_n(f;X_q^{(0,\gamma)}), f \in X_q^{(0,\gamma)}, n \in \mathbb{N}.$$

Conversely, to obtain the embedding $X_p^{(\alpha,\gamma+1/q)} \hookrightarrow (X_q^{(0,\gamma)})_p^{\alpha}$ we use the representation theorem for $X_p^{(\alpha,\gamma+1/q)}$.

Let $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma \geqslant -1/q$. The space $Z_{\alpha,p,\gamma,q}$ is formed by all $\xi \in \ell_\infty$ for which

$$\left( \sum_{n=1}^{\infty} \left[ (1 + \log n)^{\gamma} \left( \sum_{j=n}^{\infty} (j^{\alpha} \xi_j^*)^p j^{-1} \right)^{1/p} \right]^q n^{-1} \right)^{1/q} < \infty.$$

Let $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma \geqslant -1/q$. The space $Z_{\alpha,p,\gamma,q}$ is formed by all $\xi \in \ell_\infty$ for which

$$\left( \sum_{n=1}^{\infty} \left[ (1 + \log n)^{\gamma} \left( \sum_{j=n}^{\infty} (j^{\alpha} \xi_j^*)^p j^{-1} \right)^{1/p} \right]^q n^{-1} \right)^{1/q} < \infty.$$

When $q = 1, Z_{\alpha,p,\gamma,1}$ is a small Lorentz sequence space.

▻ A. Fiorenza, G.E. Karadzhov, Z. Anal. Anwend. 23 (2004) 657–681.

Let $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma \geqslant -1/q$. The space $Z_{\alpha,p,\gamma,q}$ is formed by all $\xi \in \ell_\infty$ for which

$$\left( \sum_{n=1}^{\infty} \left[ (1+\log n)^{\gamma} \left( \sum_{j=n}^{\infty} (j^{\alpha} \xi_j^*)^p j^{-1} \right)^{1/p} \right]^q n^{-1} \right)^{1/q} < \infty.$$

THEOREM.- Suppose that $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma \geqslant -1/q$. Then we have with equivalence of quasi-norms

$$(X_p^{\alpha})_q^{(0,\gamma)} = X_q^{(\alpha,\gamma)} \cap \{f \in X : (E_n(f)) \in Z_{\alpha,p,\gamma,q}\}.$$

Let $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma \geqslant -1/q$. The space $Z_{\alpha,p,\gamma,q}$ is formed by all $\xi \in \ell_\infty$ for which

$$\left( \sum_{n=1}^{\infty} \left[ (1 + \log n)^{\gamma} \left( \sum_{j=n}^{\infty} (j^{\alpha} \xi_j^*)^p j^{-1} \right)^{1/p} \right]^q n^{-1} \right)^{1/q} < \infty.$$

THEOREM.- Suppose that $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma \geqslant -1/q$. Then we have with equivalence of quasi-norms

$$(X_p^{\alpha})_q^{(0,\gamma)} = X_q^{(\alpha,\gamma)} \cap \{f \in X : (E_n(f)) \in Z_{\alpha,p,\gamma,q}\}.$$

In general, $(X_p^{\alpha})_q^{(0,\gamma)} \neq X_q^{(\alpha,\delta)}$.

Let $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma \geqslant -1/q$. The space $Z_{\alpha,p,\gamma,q}$ is formed by all $\xi \in \ell_\infty$ for which

$$\left( \sum_{n=1}^{\infty} \left[ (1+\log n)^{\gamma} \left( \sum_{j=n}^{\infty} (j^{\alpha} \xi_j^*)^p j^{-1} \right)^{1/p} \right]^q n^{-1} \right)^{1/q} < \infty.$$

THEOREM.- Suppose that $\alpha > 0, 0 < p, q \leqslant \infty$ and $\gamma \geqslant -1/q$. Then we have with equivalence of quasi-norms

$$(X_p^{\alpha})_q^{(0,\gamma)} = X_q^{(\alpha,\gamma)} \cap \{f \in X : (E_n(f)) \in Z_{\alpha,p,\gamma,q}\}.$$

In general, $(X_p^{\alpha})_q^{(0,\gamma)} \neq X_q^{(\alpha,\delta)}$. The following sharp embeddings hold

$$X_q^{(\alpha,\gamma+1/\min\{p,q\})} \hookrightarrow (X_p^{\alpha})_q^{(0,\gamma)} \hookrightarrow X_q^{(\alpha,\gamma+1/\max\{p,q\})}.$$

In particular, $(X_q^{\alpha})_q^{(0,\gamma)} = X_q^{(\alpha,\gamma+1/q)}$.

# Relation between smoothness of $D^k f$ and $f$

Let $\alpha, \gamma \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. Let $(\varphi_j)_{j \in \mathbb{N}_0}$ be a dyadic resolution of unity. The Besov space $B^{\alpha,\gamma}_{p,q}(\mathbb{T})$ is formed by all $f \in \mathcal{D}'(\mathbb{T})$ such that

$$\|f\|_{B^{\alpha,\gamma}_{p,q}(\mathbb{T})} = \left( \sum_{j=0}^{\infty} (2^{j\alpha}(1+j)^{\gamma} \|\mathfrak{F}^{-1}(\varphi_j \mathfrak{F} f)\|_{L^p(\mathbb{T})})^q \right)^{1/q} < \infty.$$

Here $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ denotes the Fourier transform and inverse Fourier transform, respectively.

# Relation between smoothness of $D^k f$ and $f$

Let $\alpha, \gamma \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. Let $(\varphi_j)_{j \in \mathbb{N}_0}$ be a dyadic resolution of unity. The Besov space $B^{\alpha,\gamma}_{p,q}(\mathbb{T})$ is formed by all $f \in \mathcal{D}'(\mathbb{T})$ such that

$$\|f\|_{B^{\alpha,\gamma}_{p,q}(\mathbb{T})} = \left( \sum_{j=0}^{\infty} (2^{j\alpha}(1+j)^{\gamma} \|\mathfrak{F}^{-1}(\varphi_j \mathfrak{F} f)\|_{L^p(\mathbb{T})})^q \right)^{1/q} < \infty.$$

Here $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ denotes the Fourier transform and inverse Fourier transform, respectively.

Let $k \in \mathbb{N}$. It holds that if $f \in B^{k,\gamma}_{p,q}(\mathbb{T})$ then $D^k f \in B^{0,\gamma}_{p,q}(\mathbb{T})$.

Let $\alpha \geqslant 0, \gamma \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. The Besov space $\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$ is formed by all $f \in L^p(\mathbb{T})$ such that

$$\|f\|_{\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})} + \left(\int_0^1 (t^{-\alpha}(1-\log t)^{\gamma}\omega_M(f,t)_p)^q \frac{dt}{t}\right)^{1/q} < \infty$$

where $M \in \mathbb{N}$ with $M > \alpha$.

Let $\alpha \geqslant 0, \gamma \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. The Besov space $\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$ is formed by all $f \in L^p(\mathbb{T})$ such that

$$\|f\|_{\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})} + \left(\int_0^1 (t^{-\alpha}(1 - \log t)^{\gamma} \omega_M(f,t)_p)^q \frac{dt}{t}\right)^{1/q} < \infty$$

where $M \in \mathbb{N}$ with $M > \alpha$.

We have that $(L^p(\mathbb{T}))_q^{(\alpha,\gamma)} = \mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$.

Let $\alpha \geqslant 0, \gamma \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. The Besov space $\mathbf{B}^{\alpha,\gamma}_{p,q}(\mathbb{T})$ is formed by all $f \in L^p(\mathbb{T})$ such that

$$\|f\|_{\mathbf{B}^{\alpha,\gamma}_{p,q}(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})} + \left( \int_0^1 (t^{-\alpha}(1 - \log t)^\gamma \omega_M(f,t)_p)^q \frac{dt}{t} \right)^{1/q} < \infty$$

where $M \in \mathbb{N}$ with $M > \alpha$.

We have that $(L^p(\mathbb{T}))^{(\alpha,\gamma)}_q = \mathbf{B}^{\alpha,\gamma}_{p,q}(\mathbb{T})$.

It holds that $B^{\alpha,\gamma}_{p,q}(\mathbb{T}) = \mathbf{B}^{\alpha,\gamma}_{p,q}(\mathbb{T})$ if $\alpha > 0$.

Let $\alpha \geqslant 0, \gamma \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. The Besov space $\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$ is formed by all $f \in L^p(\mathbb{T})$ such that

$$\|f\|_{\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})} + \left(\int_0^1 (t^{-\alpha}(1-\log t)^\gamma \omega_M(f,t)_p)^q \frac{dt}{t}\right)^{1/q} < \infty$$

where $M \in \mathbb{N}$ with $M > \alpha$.

We have that $(L^p(\mathbb{T}))_q^{(\alpha,\gamma)} = \mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$.

It holds that $B_{p,q}^{\alpha,\gamma}(\mathbb{T}) = \mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$ if $\alpha > 0$. However, $B_{p,q}^{0,\gamma}(\mathbb{T}) \neq \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$.

Let $\alpha \geqslant 0, \gamma \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. The Besov space $\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$ is formed by all $f \in L^p(\mathbb{T})$ such that

$$\|f\|_{\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})} + \left(\int_0^1 (t^{-\alpha}(1-\log t)^\gamma \omega_M(f,t)_p)^q \frac{dt}{t}\right)^{1/q} < \infty$$

where $M \in \mathbb{N}$ with $M > \alpha$.

We have that $(L^p(\mathbb{T}))_q^{(\alpha,\gamma)} = \mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$.

It holds that $B_{p,q}^{\alpha,\gamma}(\mathbb{T}) = \mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$ if $\alpha > 0$. However, $B_{p,q}^{0,\gamma}(\mathbb{T}) \neq \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$.

▻ F. Cobos, O. Domínguez, J. Math. Anal. Appl. 425 (2015) 71–84.

For $\gamma > -1/2$, we have that $B_{2,2}^{0,\gamma+1/2} = \mathbf{B}_{2,2}^{0,\gamma}$.

Let $\alpha \geqslant 0, \gamma \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. The Besov space $\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$ is formed by all $f \in L^p(\mathbb{T})$ such that

$$\|f\|_{\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})} + \left(\int_0^1 (t^{-\alpha}(1-\log t)^{\gamma}\omega_M(f,t)_p)^q \frac{dt}{t}\right)^{1/q} < \infty$$

where $M \in \mathbb{N}$ with $M > \alpha$.

We have that $(L^p(\mathbb{T}))_q^{(\alpha,\gamma)} = \mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$.

It holds that $B_{p,q}^{\alpha,\gamma}(\mathbb{T}) = \mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$ if $\alpha > 0$. However, $B_{p,q}^{0,\gamma}(\mathbb{T}) \neq \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$.

▻ F. Cobos, O. Domínguez, J. Math. Anal. Appl. 425 (2015) 71–84.

For $\gamma > -1/2$, we have that $B_{2,2}^{0,\gamma+1/2} = \mathbf{B}_{2,2}^{0,\gamma}$. As a consequence, in order to have $D^k f \in \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$ we need that $f \in \mathbf{B}_{p,q}^{k,\gamma+\delta}(\mathbb{T})$ for some $\delta > 0$.

Let $\alpha \geqslant 0, \gamma \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. The Besov space $\mathbf{B}^{\alpha,\gamma}_{p,q}(\mathbb{T})$ is formed by all $f \in L^p(\mathbb{T})$ such that

$$\|f\|_{\mathbf{B}^{\alpha,\gamma}_{p,q}(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})} + \left(\int_0^1 (t^{-\alpha}(1-\log t)^{\gamma}\omega_M(f,t)_p)^q \frac{dt}{t}\right)^{1/q} < \infty$$

where $M \in \mathbb{N}$ with $M > \alpha$.

We have that $(L^p(\mathbb{T}))_q^{(\alpha,\gamma)} = \mathbf{B}^{\alpha,\gamma}_{p,q}(\mathbb{T})$.

It holds that $B^{\alpha,\gamma}_{p,q}(\mathbb{T}) = \mathbf{B}^{\alpha,\gamma}_{p,q}(\mathbb{T})$ if $\alpha > 0$. However, $B^{0,\gamma}_{p,q}(\mathbb{T}) \neq \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$.

▻ F. Cobos, O. Domínguez, J. Math. Anal. Appl. 425 (2015) 71–84.

For $\gamma > -1/2$, we have that $B^{0,\gamma+1/2}_{2,2} = \mathbf{B}^{0,\gamma}_{2,2}$. As a consequence, in order to have $D^k f \in \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$ we need that $f \in \mathbf{B}^{k,\gamma+\delta}_{p,q}(\mathbb{T})$ for some $\delta > 0$.

- (DeVore, Riemenschneider, Sharpley) If $f \in \mathbf{B}^{k,\gamma+1}_{p,q}(\mathbb{T})$ then $D^k f \in \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$.

THEOREM.- Let $1 < p < \infty, 0 < q \leqslant \infty$ and $\gamma > -1/q$. If $f \in \mathbf{B}_{p,q}^{k,\gamma+1/\min\{2,p,q\}}(\mathbb{T})$ then $D^k f \in \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$.

THEOREM.- Let $1 < p < \infty, 0 < q \leqslant \infty$ and $\gamma > -1/q$. If $f \in \mathbf{B}_{p,q}^{k,\gamma+1/\min\{2,p,q\}}(\mathbb{T})$ then $D^k f \in \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$.

PROOF.- Since the operator $D^k : W_p^k(\mathbb{T}) \longrightarrow L^p(\mathbb{T})$ is bounded, then

$$D^k : (W_p^k(\mathbb{T}))_q^{(0,\gamma)} \longrightarrow (L^p(\mathbb{T}))_q^{(0,\gamma)} = \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T}).$$

THEOREM.- Let $1 < p < \infty, 0 < q \leqslant \infty$ and $\gamma > -1/q$. If $f \in \mathbf{B}^{k,\gamma+1/\min\{2,p,q\}}_{p,q}(\mathbb{T})$ then $D^k f \in \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$.

PROOF.- Since the operator $D^k : W^k_p(\mathbb{T}) \longrightarrow L^p(\mathbb{T})$ is bounded, then

$$D^k : (W^k_p(\mathbb{T}))^{(0,\gamma)}_q \longrightarrow (L^p(\mathbb{T}))^{(0,\gamma)}_q = \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T}).$$

Applying reiteration constructions we derive that

$$\begin{aligned}(W^k_p(\mathbb{T}))^{(0,\gamma)}_q &\hookleftarrow (\mathbf{B}^k_{p,\min\{p,2\}}(\mathbb{T}))^{(0,\gamma)}_q = ((L^p(\mathbb{T}))^k_{\min\{p,2\}})^{(0,\gamma)}_q \\ &\hookleftarrow (L^p(\mathbb{T}))^{(k,\gamma+1/\min\{2,p,q\})}_q = \mathbf{B}^{k,\gamma+1/\min\{2,p,q\}}_{p,q}(\mathbb{T}).\end{aligned}$$

THEOREM.- Let $1 < p < \infty, 0 < q \leqslant \infty$ and $\gamma > -1/q$. If $f \in \mathbf{B}_{p,q}^{k,\gamma+1/\min\{2,p,q\}}(\mathbb{T})$ then $D^k f \in \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$.

PROOF.- Since the operator $D^k : W_p^k(\mathbb{T}) \longrightarrow L^p(\mathbb{T})$ is bounded, then

$$D^k : (W_p^k(\mathbb{T}))_q^{(0,\gamma)} \longrightarrow (L^p(\mathbb{T}))_q^{(0,\gamma)} = \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T}).$$

Applying reiteration constructions we derive that

$$\begin{aligned}(W_p^k(\mathbb{T}))_q^{(0,\gamma)} &\hookleftarrow (\mathbf{B}_{p,\min\{p,2\}}^k(\mathbb{T}))_q^{(0,\gamma)} = ((L^p(\mathbb{T}))_{\min\{p,2\}}^k)_q^{(0,\gamma)} \\ &\hookleftarrow (L^p(\mathbb{T}))_q^{(k,\gamma+1/\min\{2,p,q\})} = \mathbf{B}_{p,q}^{k,\gamma+1/\min\{2,p,q\}}(\mathbb{T}).\end{aligned}$$

REMARK.- The previous result is the best possible.

▷ F. Cobos, O. Domínguez, preprint (2015).