One-scale H-measures, variants and applications

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Linköping, 17^{th} June, 2015

Joint work with Nenad Antonić and Martin Lazar

WeConMApp

Introduction

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If the defect measure is not trivial we need another objects to determine all the properties of the sequence:

- H-measures
- semiclassical measures
- ...

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Measure $\boldsymbol{\mu}_H$ *we call the H-measure corresponding to the (sub)sequence* (u_n).

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[T1] Luc Tartar: *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, Proceedings of the Royal Society of Edinburgh*, **115A** (1990) 193–230.

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Semiclassical measures

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$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{sc}^{(\omega_n)} = \mathbf{0} \quad \& \quad (\mathsf{u}_n)\ \textit{is}\ (\omega_n) - \textit{oscillatory}\ .$$

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Definition

(u_n) is (ω_n)-oscillatory if
$(\forall\,\varphi\in\mathrm{C}^\infty_c(\Omega))\quad \lim_{R\to\infty}\limsup_n\int_{|\boldsymbol{\xi}|\geqslant\frac{R}{\omega_n}}|\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}=0\,.$

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(u_n) from $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$, $\omega_n \to 0^+$,

$$\mathbf{W}_n(\mathbf{x},\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\xi}} \mathsf{u}_n\Big(\mathbf{x}+\frac{\omega_n \mathbf{y}}{2}\Big) \otimes \mathsf{u}_n\Big(\mathbf{x}-\frac{\omega_n \mathbf{y}}{2}\Big)\, d\mathbf{y}$$

Theorem

If $\mathsf{u}_n \rightharpoonup \mathsf{u}$ *in* $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, *then there exists* $(\mathsf{u}_{n'})$ *such that*

$$\mathbf{W}_{n'} \xrightarrow{\mathcal{S}'} \boldsymbol{\mu}_{sc}^{(\omega_{n'})} .$$

[G2] Patrick Gérard: *Mesures semi-classiques et ondes de Bloch, Sem. EDP 1990–91 (exp. 16)*, (1991)

[LP] Pierre Louis Lions, Thierry Paul: *Sur les measures de Wigner, Revista Mat. Iberoamericana* **9,** (1993) 553-618

Example: Oscillations - one characteristic length

$\alpha > 0$, $\mathsf{k} \in \mathbf{Z}^d \setminus \{\mathsf{0}\}$,

$$u_n(\mathbf{x}) := e^{2\pi i n^{\alpha} \mathsf{k}\cdot\mathbf{x}} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,,\ n \to \infty$$

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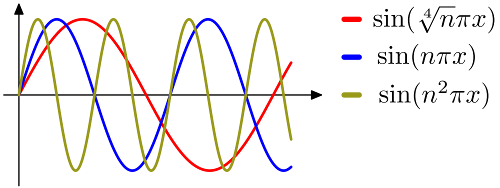

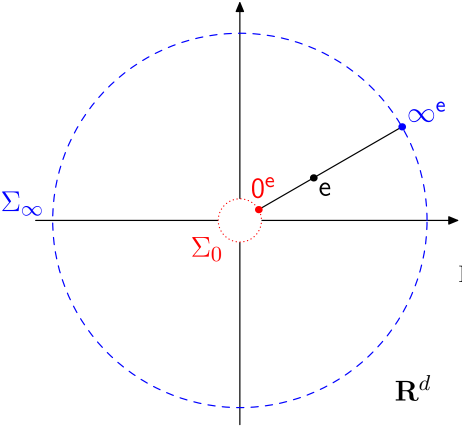

$$\Sigma_0 := \{0^{\mathsf{e}} \ : \ \mathsf{e} \in \mathrm{S}^{d-1}\}$$

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$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d \setminus \{0\} \cup \Sigma_0 \cup \Sigma_\infty$$

Corollary

a) $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
b) $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\psi \circ \boldsymbol{\pi} \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *where* $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

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[T2] Luc Tartar: *The general theory of homogenization: A personalized introduction, Springer* (2009)

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[AEL] Nenad Antonić, M.E., Martin Lazar: *Localisation principle for one-scale H-measures*, submitted (arXiv).

Tartar's approach:

- $\mathsf{v}_n(\mathbf{x}, x^{d+1}) := \mathsf{u}_n(\mathbf{x})e^{\frac{2\pi i x^{d+1}}{\omega_n}} \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega \times \mathbf{R}; \mathbf{C}^r)$
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- $\boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}$ is obtained from $\boldsymbol{\nu}_H$ (suitable projection in x^{d+1} and ξ_{d+1})

Idea of the proof

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Our approach:

- commutation lemma

Lemma

Let $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, and denote $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. Then the commutator can be expressed as a sum

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = \tilde{C}_n + K \,,$$

where K is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$, while $\tilde{C}_n \longrightarrow 0$ in the operator norm on $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$.

- standard procedure ((a variant of) kernel lemma, separability...)

Theorem

a) $\boldsymbol{\mu}^*_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}\ , \qquad \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \geqslant \mathbf{0}$

c) $\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}$

d) $\mathrm{tr}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}(\Omega \times \Sigma_\infty) = 0 \iff (\mathsf{u}_n)$ *is* (ω_n) *− oscillatory*

Some properties of $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$

Theorem

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Theorem

$\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$, $\psi \in \mathrm{C}_0(\mathbf{R}^d)$, $\tilde{\psi} \in \mathrm{C}(\mathrm{S}^{d-1})$, $\omega_n \to 0^+$,

a) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \rangle = \langle \boldsymbol{\mu}^{(\omega_n)}_{sc}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \rangle$,

b) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar{\varphi}_2 \boxtimes \tilde{\psi} \circ \boldsymbol{\pi} \rangle = \langle \boldsymbol{\mu}_H, \varphi_1\bar{\varphi}_2 \boxtimes \tilde{\psi} \rangle$,

where $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

Example revisited

$$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}},$$

$$\mu_H = \lambda \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$$

$$\mu_{sc}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_0 & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty \rangle \\ 0 & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

$$\mu_{\mathrm{K}_{0,\infty}}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_{0\frac{\mathsf{k}}{|\mathsf{k}|}} & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty \rangle \\ \delta_{\infty\frac{\mathsf{k}}{|\mathsf{k}|}} & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

Motivation (localisation principle for H-measures)

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M}_r(\mathbf{C}))$ and

$$\mathbf{P}\mathsf{u}_n := \sum_{|\boldsymbol{\alpha}|=m} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) \longrightarrow 0 \text{ in } \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)\,.$$

Then we have

$$\mathbf{p}\boldsymbol{\mu}_H^\top = \mathbf{0}\,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}|=m} \boldsymbol{\xi}^{\boldsymbol{\alpha}}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$ is the principle simbol of $\mathbf{P}$.

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Idea: If $d = 1$ and p is nowhere zero (e.g. elliptic operator of the second order), we know $\mu_H = 0$, and that implies $u_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

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Applications:

- compactness by compensation
- small amplitude homogenisation
- velocity averaging
- averaged control

...

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$$\mathbf{P}_n \mathsf{u}_n := \sum_{|\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,,$$

where

- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

Then we have

$$\mathbf{p} \boldsymbol{\mu}_{sc}^{\top} = \mathbf{0} \,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leqslant m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$, and $\boldsymbol{\mu}_{sc}$ is semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

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where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leqslant m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$, and $\boldsymbol{\mu}_{sc}$ is semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

Problem: $\boldsymbol{\mu}_{sc} = \mathbf{0}$ is not enough for the strong convergence!

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,,$$

where

- $l \in 0..m$
- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (\mathrm{C}(\varepsilon_n))$$

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

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Lemma

a) $(\mathrm{C}(\varepsilon_n))$ *is equivalent to*

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + |\boldsymbol{\xi}|^l + \varepsilon_n^{m-l} |\boldsymbol{\xi}|^m} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)\,.$$

b) $(\exists\, k \in l..m)$ $\mathsf{f}_n \longrightarrow 0$ *in* $\mathrm{H}^{-k}_{\mathrm{loc}}(\Omega; \mathbf{C}^r) \quad \Longrightarrow \quad (\varepsilon_n^{k-l} \mathsf{f}_n)$ *satisfies* $(\mathrm{C}(\varepsilon_n))$.

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,. \qquad (\mathrm{C}(\varepsilon_n))$$

Theorem (Tartar (2009))

Under previous assumptions and $l=1$, 1-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) corresponding to (u_n) satisfies

$$\operatorname{supp}\left(\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top\right) \subseteq \Omega\times\Sigma_0\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{1\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Theorem

$\varepsilon_n > 0$ *bounded* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ *and*

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}_n \mathsf{u}_n) = \mathsf{f}_n\,,$$

where $\mathbf{A}^{\boldsymbol{\alpha}}_n \in \mathrm{C}(\Omega;\mathrm{M}_r(\mathbf{C}))$, $\mathbf{A}^{\boldsymbol{\alpha}}_n \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ *uniformly on compact sets, and* $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ *satisfies* ($\mathrm{C}(\varepsilon_n)$).
Then for $\omega_n \to 0^+$ *such that* $c := \lim_n \frac{\varepsilon_n}{\omega_n} \in [0,\infty]$, *corresponding 1-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ω_n) *satisfies*

$$\mathbf{p}\boldsymbol{\mu}^{\top}_{\mathrm{K}_{0,\infty}} = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c=0 \\ \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m}(2\pi i c)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c\in\langle 0,\infty\rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c=\infty \end{cases}$$

Theorem (cont.)

Moreover, if there exists $\varepsilon_0 > 0$ *such that* $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, *we can take*

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) \,.$$

Localisation principle - final generalisation

Theorem (cont.)

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As a corollary from the previous theorem we can derive localisation principles for H-measures and semiclassical measures.

Example: equations with characteristic length (1/2)

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

$$\begin{cases} u_n^1 + \varepsilon_n \partial_{x_1}(a_1 u_n^1) = f_n^1 \\ u_n^2 + \varepsilon_n \partial_{x_2}(a_2 u_n^2) = f_n^2 \end{cases},$$

where $\varepsilon_n \to 0^+$, $\mathsf{f}_n := (f_n^1,\ f_n^2) \in \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies $(\mathrm{C}(\varepsilon_n))$ (with $l = 0, m = 1$), while $a_1, a_2 \in \mathrm{C}(\Omega; \mathbf{R})$, $a_1, a_2 \neq 0$ everywhere.

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

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By the localisation principle for one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) (i.e. $c = 1$) associated to (u_n) we get the relation

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2\pi i \xi_1}{1+|\boldsymbol{\xi}|} \begin{bmatrix} a_1(\mathbf{x}) & 0 \\ 0 & 0 \end{bmatrix} + \frac{2\pi i \xi_2}{1+|\boldsymbol{\xi}|} \begin{bmatrix} 0 & 0 \\ 0 & a_2(\mathbf{x}) \end{bmatrix} \right) \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0}\,,$$

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whose $(1,1)$ component reads

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} + i \frac{2\pi \xi_1}{1+|\boldsymbol{\xi}|} a_1(\mathbf{x}) \right) \mu^{11}_{\mathrm{K}_{0,\infty}} = 0\,,$$

hence

$$\frac{1}{1+|\boldsymbol{\xi}|} \mu^{11}_{\mathrm{K}_{0,\infty}} = 0\,, \qquad \frac{\xi_1}{1+|\boldsymbol{\xi}|} \mu^{11}_{\mathrm{K}_{0,\infty}} = 0$$

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

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whose $(1,1)$ component reads

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} + i \frac{2\pi \xi_1}{1+|\boldsymbol{\xi}|} a_1(\mathbf{x}) \right) \mu_{\mathrm{K}_{0,\infty}}^{11} = 0\,,$$

hence

$$\operatorname{supp} \mu_{\mathrm{K}_{0,\infty}}^{11} \subseteq \Omega \times \Sigma_\infty\,, \qquad \frac{\xi_1}{1+|\boldsymbol{\xi}|} \mu_{\mathrm{K}_{0,\infty}}^{11} = 0$$

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

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$$\operatorname{supp} \mu^{11}_{\mathrm{K}_{0,\infty}} \subseteq \Omega \times \Sigma_\infty\,, \quad \operatorname{supp} \mu^{11}_{\mathrm{K}_{0,\infty}} \subseteq \Omega \times (\Sigma_0 \cup \{\xi_1 = 0\})$$

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

$$\begin{cases} u_n^1 + \varepsilon_n \partial_{x_1}(a_1 u_n^1) = f_n^1 \\ u_n^2 + \varepsilon_n \partial_{x_2}(a_2 u_n^2) = f_n^2 \end{cases},$$

where $\varepsilon_n \to 0^+$, $\mathsf{f}_n := (f_n^1,\ f_n^2) \in \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies $(\mathrm{C}(\varepsilon_n))$ (with $l = 0, m = 1$), while $a_1, a_2 \in \mathrm{C}(\Omega; \mathbf{R})$, $a_1, a_2 \neq 0$ everywhere.

By the localisation principle for one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) (i.e. $c = 1$) associated to (u_n) we get the relation

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2\pi i \xi_1}{1+|\boldsymbol{\xi}|} \begin{bmatrix} a_1(\mathbf{x}) & 0 \\ 0 & 0 \end{bmatrix} + \frac{2\pi i \xi_2}{1+|\boldsymbol{\xi}|} \begin{bmatrix} 0 & 0 \\ 0 & a_2(\mathbf{x}) \end{bmatrix} \right) \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0},$$

whose $(1,1)$ component reads

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} + i \frac{2\pi \xi_1}{1+|\boldsymbol{\xi}|} a_1(\mathbf{x}) \right) \mu_{\mathrm{K}_{0,\infty}}^{11} = 0,$$

hence

$$\operatorname{supp} \mu_{\mathrm{K}_{0,\infty}}^{11} \subseteq \Omega \times \Sigma_\infty, \quad \operatorname{supp} \mu_{\mathrm{K}_{0,\infty}}^{11} \subseteq \Omega \times (\Sigma_0 \cup \{\xi_1 = 0\})$$

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$$\operatorname{supp} \mu_{\mathrm{K}_{0,\infty}}^{11} \subseteq \Omega \times \{\infty^{(0,-1)}, \infty^{(0,1)}\}$$

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whose $(1,1)$ component reads

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} + i\frac{2\pi\xi_1}{1+|\boldsymbol{\xi}|} a_1(\mathbf{x})\right) \mu^{11}_{\mathrm{K}_{0,\infty}} = 0\,,$$

hence

$$\operatorname{supp} \mu^{11}_{\mathrm{K}_{0,\infty}} \subseteq \Omega \times \{\infty^{(0,-1)}, \infty^{(0,1)}\}$$

Example: equations with characteristic length (2/2)

Analogously, from the $(2,2)$ component we get

$$\operatorname{supp}\mu^{22}_{\mathrm{K}_{0,\infty}} \subseteq \Omega \times \{\infty^{(-1,0)}, \infty^{(1,0)}\},$$

hence $\operatorname{supp}\mu^{11}_{\mathrm{K}_{0,\infty}} \cap \operatorname{supp}\mu^{22}_{\mathrm{K}_{0,\infty}} = \emptyset$ which implies $\mu^{12}_{\mathrm{K}_{0,\infty}} = \mu^{21}_{\mathrm{K}_{0,\infty}} = 0$.

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The very definition of one-scale H-measures gives $u^1_n \bar{u^2_n} \xrightarrow{*} 0$.

This approach can be systematically generalised by introducing a variant of compensated compactness suitable for problems with characteristic length.

Let $\mathsf{u}_n \longrightarrow \mathsf{u}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ satisfy

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n\,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ in $\mathrm{C}(\Omega;\mathrm{M}_{\mathrm{q}\times\mathrm{r}}(\mathbf{C}))$, let $\varepsilon_n \to 0^+$, and $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega;\mathbf{C}^q)$ be such that for any $\varphi\in\mathrm{C}_c^\infty(\Omega)$

$$\frac{\widehat{\varphi\mathsf{f}_n}}{1+k_n}$$

is precompact in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^q)$. Furthermore, let $Q(\mathbf{x};\boldsymbol{\lambda}) := \mathbf{Q}(\mathbf{x})\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}$, where $\mathbf{Q}\in\mathrm{C}(\Omega;\mathrm{M}_{\mathrm{r}}(\mathbf{C}))$, $\mathbf{Q}^*=\mathbf{Q}$, is such that $Q(\cdot;\mathsf{u}_n)\xrightarrow{*}\nu$ in $\mathcal{M}(\Omega)$.
Then we have

a) $(\exists\, c\in[0,\infty])(\forall\,(\mathbf{x},\boldsymbol{\xi})\in\Omega\times\mathrm{K}_{0,\infty}(\mathbf{R}^d))(\forall\,\boldsymbol{\lambda}\in\Lambda_{c;\mathbf{x},\boldsymbol{\xi}})\; Q(\mathbf{x};\boldsymbol{\lambda})\geqslant 0 \implies \nu\geqslant Q(\cdot,\mathsf{u})$,

b) $(\exists\, c\in[0,\infty])(\forall\,(\mathbf{x},\boldsymbol{\xi})\in\Omega\times\mathrm{K}_{0,\infty}(\mathbf{R}^d))(\forall\,\boldsymbol{\lambda}\in\Lambda_{c;\mathbf{x},\boldsymbol{\xi}})\; Q(\mathbf{x};\boldsymbol{\lambda})= 0 \implies \nu = Q(\cdot,\mathsf{u})$,

where

$$\Lambda_{c;\mathbf{x},\boldsymbol{\xi}} := \{\boldsymbol{\lambda}\in\mathbf{C}^r : \mathbf{p}_c(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\lambda}=0\}\,,$$

and $\mathbf{p}_c$ is given as before.

Theorem

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$, $v_n \overset{*}{\rightharpoonup} 0$ *in* $\mathrm{L}^q(\mathbf{R}^d)$, $q \geqslant p'$, *and* $\omega_n \to 0^+$, *then there exist* $(u_{n'}), (v_{n'})$ *and* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}} \, d\mathbf{x} = \lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 u_{n'}) \overline{\mathcal{A}_{\bar{\psi}_{n'}}(\varphi_2 v_{n'})} \, d\mathbf{x}$$
$$= \langle \mu, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \,,$$

where $\mathcal{A}_{\psi_{n'}}(\mathsf{u}) = (\psi_{n'} \hat{\mathsf{u}})^\wedge$ *and* $\psi_{n'}(\boldsymbol{\xi}) := \psi(\varepsilon_{n'} \boldsymbol{\xi})$.

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<u>Technical difficulties:</u>

- differential structure on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$
- Hörmander-Mihlin condition for $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$
- distributions on compact set (manifold with boundary)

Thank you
for your
attention! :)