Hardy–Sobolev inequalities on general open sets

Juha Lehrbäck

based on a joint work with Antti Vähäkangas

Jyväskylän yliopisto

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1. Hardy–Sobolev inequalities

Hardy and Sobolev inequalities

Let $\Omega \subset \mathbb{R}^n$ be an open set, let $1 \leq p < n$, and denote $p^* = np/(n-p)$. Then the Sobolev inequality

$$\left(\int_\Omega |u|^{p^*}\,dx\right)^{1/p^*} \leq C\left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$$

holds for all $u \in C_0^\infty(\Omega)$.

The (p,β)-Hardy inequality, for $1 \leq p < \infty$ and $\beta \in \mathbb{R}$, reads as

$$\int_\Omega |u|^p\,\delta_{\partial\Omega}^{\beta-p}\,dx \leq C\int_\Omega |\nabla u|^p\,\delta_{\partial\Omega}^{\beta}\,dx,$$

where $\delta_{\partial\Omega}(x) = \operatorname{dist}(x,\partial\Omega)$. If there is $C > 0$ such that this holds for all $u \in C_0^\infty(\Omega)$, we say that Ω admits a (p,β)-Hardy inequality.

These inequalities are well-known tools in the study of function spaces, e.g. (weighted) Sobolev spaces, and have applications in the theory of PDE's.

In this talk, we are interested in the following inequalities forming a natural interpolating scale in between the (weighted) Sobolev inequalities and the (weighted) Hardy inequalities.

An open set $\Omega \subsetneq \mathbb{R}^n$ *admits a* (q, p, β)*-Hardy–Sobolev inequality* if there is $C > 0$ such that

$$\left(\int_\Omega |u|^q \, \delta_{\partial\Omega}^{(q/p)(n-p+\beta)-n} \, dx \right)^{1/q} \leq C \left(\int_\Omega |\nabla u|^p \, \delta_{\partial\Omega}^{\beta} \, dx \right)^{1/p} \qquad (1)$$

for all $u \in C_0^\infty(\Omega)$.

The Sobolev inequality is the case $q = p^* = np/(n-p)$, $\beta = 0$ in (1).

The weighted (p, β)-Hardy inequality is the case $q = p$ in (1).

Some history

When $E \subset \mathbb{R}^n$ is an m-dimensional subspace, $1 \le m \le n-1$, $\Omega = \mathbb{R}^n \setminus E$, and $m < \frac{q}{p}(n - p + \beta)$, the global version of the (q, p, β)-Hardy–Sobolev inequality (for all $f \in C_0^\infty(\mathbb{R}^n)$) is due to Maz'ya [M, 1985].

Badiale and Tarantello [BT, 2002] (essentially) rediscovered Maz'ya's result for $\beta = 0$, and applied this to study the properties of the solutions of certain elliptic PDE's related to the dynamics of galaxies.

For $m = 0$, i.e. $E = \{0\}$, the corresponding Hardy–Sobolev inequality is known as Caffarelli–Kohn–Nirenberg inequality, since this case first appeared in [CKN, 1984]

"Interpolation"

Hardy–Sobolev inequalities can be obtained from the (weighted) Hardy inequality with the help of the (unweighted) Sobolev inequality:

Theorem (LV, 2015)

Assume that $1 \leq p < n$ *and* $\beta \in \mathbb{R}$. *If* Ω *admits a* (p, p, β)*-Hardy–Sobolev inequality (i.e., a* (p, β)*-Hardy inequality), then* Ω *admits* (q, p, β)*-Hardy–Sobolev inequalities for all exponents* $p \leq q \leq p^*$.

Proof of the interpolation theorem

Step 1: (p,p,β)-HS $\implies$ (p^*,p,β)-HS (weighted Sobolev).
Let $u \in C_0^\infty(\Omega)$ and denote $g = |u|\delta_{\partial\Omega}^{\beta/p} \in \mathrm{Lip}_0(\Omega)$. Then using the Sobolev inequality for g and the (p,p,β)-HS inequality for u we obtain

$$\begin{aligned}\left(\int_\Omega |u|^{p^*}\delta_{\partial\Omega}^{\frac{n\beta}{n-p}}\right)^{1/p^*} &= \left(\int_\Omega |g|^{p^*}\right)^{1/p^*} \\ &\lesssim \left(\int_\Omega |\nabla g|^p\right)^{1/p} \lesssim \left(\int_\Omega |\nabla u|^p \delta_{\partial\Omega}^{\beta}\right)^{1/p} + \left(\int_\Omega |u|^p \delta_{\partial\Omega}^{\beta-p}\right)^{1/p} \\ &\lesssim \left(\int_\Omega |\nabla u|^p \delta_{\partial\Omega}^{\beta}\right)^{1/p}\end{aligned}$$

Step 2: The (p,p,β)- and (p^*,p,β)-HS inequalities and Hölder's inequality yield (q,p,β)-HS inequalities for all $p \le q \le p^*$:

$$\left(\int_\Omega |u|^q\,\delta_{\partial\Omega}^{(q/p)(n-p+\beta)-n}\right)^{1/q} \le \left(\int_\Omega |u|^p\,\delta_{\partial\Omega}^{\beta-p}\right)^{\frac{1}{q\alpha}}\left(\int_\Omega |u|^{p^*}\,\delta_{\partial\Omega}^{\frac{n\beta}{n-p}}\right)^{\frac{1}{q\alpha'}}.$$

2. Assouad dimensions

Assouad dimensions

Let $E \subset \mathbb{R}^n$. Consider all exponents $\lambda \geq 0$ for which there is $C \geq 1$ such that $E \cap B(w, R)$ can be covered by *at most* $C(r/R)^{-\lambda}$ balls of radius r for all $0 < r < R < \operatorname{diam}(E)$ and $w \in E$.

The infimum of such exponents λ is the *(upper) Assouad dimension* $\overline{\dim}_{\mathrm{A}}(E)$.

Conversely: consider all $\lambda \geq 0$ for which there is $c > 0$ such that if $0 < r < R < \operatorname{diam}(E)$, then for every $w \in E$ *at least* $c(r/R)^{-\lambda}$ balls of radius r are needed to cover $E \cap B(w, R)$.

The supremum of all such λ is the *lower Assouad dimension* $\underline{\dim}_{\mathrm{A}}(E)$.

Some comments on Assouad dimensions

(Upper) Assouad dimension was introduced by P. Assouad around 1980 in connection to bi-Lipschitz embedding problem between metric and Euclidean spaces. However, equivalent (or closely related) concepts have appeared under different names, e.g. *(uniform) metric dimension*, some dating back (at least) to [Bouligand 1928]. See [Luukkainen 1998] for a nice account on the basic properties of (upper) Assouad dimension as well as some historical comments.

Lower Assouad dimension has (essentially) appeared under names *lower dimension*, *minimal dimensional number*, and *uniformity dimension*. Some basic properties of this are recently established in [Fraser 2014] and [KLV 2013].

Minkowski and Assouad

Once again:
$\overline{\dim}_A(E)$ is the infimum of $\lambda \geq 0$ s.t. $E \cap B(w, R)$ can (always) be covered by at most $C(r/R)^{-\lambda}$ balls of radius $0 < r < R < \operatorname{diam}(E)$

$\underline{\dim}_A(E)$ is the supremum of $\lambda \geq 0$ s.t. (always) at least $C(r/R)^{-\lambda}$ balls of radius $0 < r < R < \operatorname{diam}(E)$ are needed to cover $E \cap B(w, R)$

For comparison, recall the *upper and lower Minkowski dimensions* of a compact $E \subset \mathbb{R}^n$:

$\overline{\dim}_M(E)$ is the infimum of $\lambda \geq 0$ s.t. E can be covered by at most $Cr^{-\lambda}$ balls of radius $0 < r < \operatorname{diam}(E)$

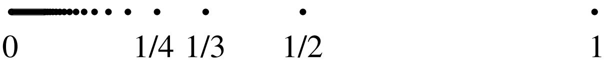
$\underline{\dim}_M(E)$ is the supremum of $\lambda \geq 0$ s.t. at least $Cr^{-\lambda}$ balls of radius $0 < r < \operatorname{diam}(E)$ are needed to cover E.

Thus $\quad \underline{\dim}_A(E) \leq \underline{\dim}_M(E) \leq \overline{\dim}_M(E) \leq \overline{\dim}_A(E).$

Examples (1)

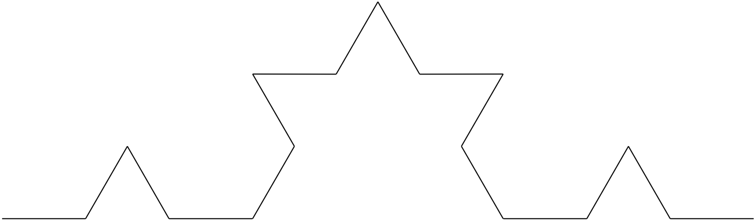
General idea: Assouad dimensions reflect the 'extreme' behavior of sets and take into account all scales $0 < r < d(E)$.

- If $E = \{0\} \cup [1,2] \subset \mathbb{R}$, then $\underline{\dim}_{\mathrm{A}}(E) = 0$ and $\overline{\dim}_{\mathrm{A}}(E) = 1$ $(\underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E) = 1)$.
- $\underline{\dim}_{\mathrm{A}}(\mathbb{Z}) = 0$ and $\overline{\dim}_{\mathrm{A}}(\mathbb{Z}) = 1$.
- If $E = \{(1/j, 0, \ldots, 0) : j \in \mathbb{N}\} \cup \{(0, 0, \ldots, 0)\} \subset \mathbb{R}^n$, then then $\underline{\dim}_{\mathrm{A}}(E) = 0$ and $\overline{\dim}_{\mathrm{A}}(E) = 1$ $(\underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E) = 1/2)$.

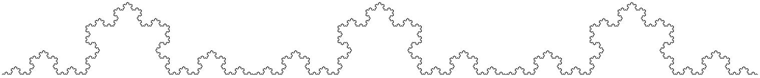


Examples (2)

- If $S \subset \mathbb{R}^2$ is an unbounded, locally rectifiable von Koch snowflake -type curve consisting of unit intervals, then $\underline{\dim}_{\mathsf{A}}(S) = 1$ and $\overline{\dim}_{\mathsf{A}}(E) = \log 4 / \log 3$ (flat on small scales, fractal on large scales)



- If $S \subset \mathbb{R}^2$ consists of infinitely many copies of the usual (fractal) von Koch snowflake curve, laid side by side, then $\underline{\dim}_{\mathsf{A}}(S) = 1$ and $\overline{\dim}_{\mathsf{A}}(E) = \log 4 / \log 3$ (fractal on small scales, flat on large scales).



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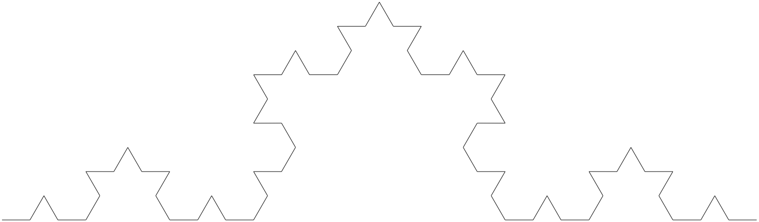

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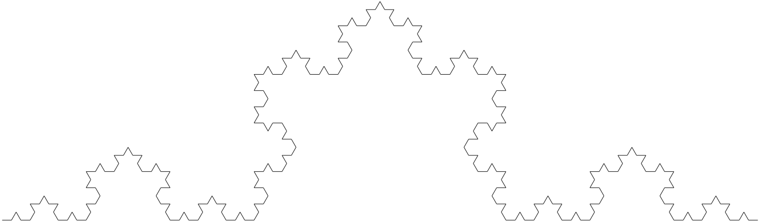

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Hausdorff and lower Assouad

Recall that the *Hausdorff (ϱ-)content* of dimension λ, for $E \subset \mathbb{R}^n$, is

$$\mathcal{H}^{\lambda}_{\varrho}(E) = \inf \Big\{ \sum_k r_k^{\lambda} : E \subset \bigcup_k B(x_k, r_k),\ x_k \in E,\ 0 < r_k \le \varrho \Big\}.$$

The λ-*Hausdorff measure* of E is $\mathcal{H}^{\lambda}(E) = \lim_{\varrho \to 0} \mathcal{H}^{\lambda}_{\varrho}(E)$ and the *Hausdorff dimension* of E is $\dim_{\mathrm{H}}(A) = \inf\{\lambda \ge 0 : \mathcal{H}^{\lambda}(A) = 0\}$.

It can be shown that if $E \subset \mathbb{R}^n$ is closed, then $\underline{\dim}_{\mathrm{A}}(E) \le \dim_{\mathrm{H}}(E \cap B)$ for all balls B centered at E. (However, e.g. $\underline{\dim}_{\mathrm{A}}(\mathbb{Q}) = 1$ but $\dim_{\mathrm{H}}(\mathbb{Q}) = 0$.)

The proof of this is based on the fact that for each $0 < t < \underline{\dim}_{\mathrm{A}}(E)$

$$\mathcal{H}^{t}_{\infty}\big(E \cap B(w, r)\big) \ge c r^t \quad \text{for all } w \in E,\ 0 < r < \operatorname{diam}(E). \qquad (2)$$

In fact, for closed $E \subset \mathbb{R}^n$ we have $\underline{\dim}_{\mathrm{A}}(E) = \sup\{t \ge 0 : (2) \text{ holds}\}$. This links $\underline{\dim}_{\mathrm{A}}$ to *uniform fatness* and hence to potential theory.

3. Results

Sufficient conditions

The following sufficient condition holds for the (p, β)-Hardy inequality.

Theorem (L, 2014)

Let $1 < p < \infty$ and $\beta < p - 1$, and let $\Omega \subset \mathbb{R}^n$ be an open set. If

$$\overline{\dim}_A(\Omega^c) < n - p + \beta \quad \text{or} \quad \underline{\dim}_A(\Omega^c) > n - p + \beta,$$

then Ω admits a (p, β)-Hardy inequality;
in the latter case, if Ω is unbounded, then also Ω^c has to be unbounded.

The first conditon has been essentially known in $\mathbb{R}^n$ in the case $\beta = 0$ by [Aikawa 1991] and [Koskela–Zhong 2003], and for general β under additional geometric assumptions [L. 2008].
The second condition, for $\beta = 0$, is a reformulation of the well-known sufficient condition using *uniform p-fatness*.

Sufficient conditions

From the interpolation theorem we obtain the corresponding result for Hardy–Sobolev inequalities for all $p \le q \le p^*$.

Theorem (LV, 2015)

Let $1 < p < \infty$ and $\beta < p-1$, and let $\Omega \subset \mathbb{R}^n$ be an open set. If

$$\overline{\dim}_A(\Omega^c) < n - p + \beta \quad \textit{or} \quad \underline{\dim}_A(\Omega^c) > n - p + \beta,$$

then Ω admits a (q,p,β)-Hardy–Sobolev inequality for all $p \le q \le p^*$; in the latter case, if Ω is unbounded, then also Ω^c has to be unbounded.

Sufficient conditions revisited

In the previous sufficient condition for Hardy–Sobolev inequalities the bound $\underline{\dim}_{\mathrm{A}}(\Omega^c) > n - p + \beta$ is rather sharp, but $\overline{\dim}_{\mathrm{A}}(\Omega^c) < n - p + \beta$ can be weakened when $p < q < p^*$. Also the upper bound $\beta < p - 1$ can be changed to the weaker assumption that $\overline{\dim}_{\mathrm{A}}(\Omega^c) < n - 1$:

Theorem (LV, 2015)

Let $1 \le p \le q \le np/(n-p) < \infty$ and $\beta \in \mathbb{R}$. If $\Omega \subset \mathbb{R}^n$ is an open set and

$$\overline{\dim}_{\mathrm{A}}(\Omega^c) < \min\left\{ \tfrac{q}{p}(n - p + \beta)\,,\, n-1 \right\},$$

then Ω admits a (q, p, β)-Hardy–Sobolev inequality.

The requirement $\overline{\dim}_{\mathrm{A}}(\Omega^c) < n - 1$ can not be omitted. An example is given by $\Omega = \mathbb{R}^n \setminus \partial B(0,1)$: for suitable functions $u_k \in C_0^\infty(B(0,1))$ the LHS of the (q,p,β)-HS has a positive lower bound, while the RHS tends to zero if $\beta > p - 1 = p - n + \overline{\dim}_{\mathrm{A}}(\Omega^c)$.

Horiuchi and $P(s)$-condition

The proof of the previous theorem relies heavily on the work of Horiuchi [H, 1989], who studied embeddings between weighted Sobolev spaces and hence the non-homogeneous versions of Hardy–Sobolev inequalities.

In this connection Horiuchi defined that a closed set $E \subset \mathbb{R}^n$ of zero measure satisfies condition $P(s)$, for $0 \le s \le n$, if there is $C > 0$ such that for all balls B and all numbers η_1, η_2 satisfying $0 \le \eta_1 < \eta_2 \le \operatorname{diam}(B)$,

$$|B \cap (E_{\eta_2} \setminus E_{\eta_1})| \le \begin{cases} C\eta_2^{s-1}(\eta_2 - \eta_1)\operatorname{diam}(B)^{n-s} & \text{if } 1 \le s \le n \\ C(\eta_2 - \eta_1)^s \operatorname{diam}(B)^{n-s} & \text{if } 0 \le s < 1\,. \end{cases}$$

Here $E_\eta = \{x \in \mathbb{R}^n : \delta_E(x) < \eta\}$.

Horiuchi and Assouad

Horiuchi's $P(s)$-condition is clearly related to the dimension of E, but perhaps the following characterization is not completely obvious:

Theorem (LV, 2015)

Let $E \subset \mathbb{R}^n$ be a closed set with $|E| = 0$. Then

$$\overline{\dim}_{\mathrm{A}}(E) = n - \sup\big\{0 \le s \le n \,:\, E \text{ satisfies } P(s)\big\}.$$

In particular, the $P(s)$-property holds for all $0 \le s < n - \overline{\dim}_{\mathrm{A}}(E)$.

Knowing this, we can follow Horiuchi's original ideas to prove our sufficient condition for Hardy–Sobolev inequalities.

Necessary conditions

Theorem (LV, 2015)

Assume that $1 \leq p \leq q < np/(n-p) < \infty$ *and that* $\Omega \subset \mathbb{R}^n$ *admits a* (q,p,β)*-HS inequality. If* $\beta \geq 0$ *and* $\frac{q}{p}(n-p+\beta) \neq n$*, then*

$$\overline{\dim}_{\mathrm{A}}(\Omega^c) < \tfrac{q}{p}(n-p+\beta) \quad \text{or} \quad \dim_{\mathrm{H}}(\Omega^c) \geq n-p+\beta\,.$$

If $\beta < 0$ *and* Ω^c *is compact and porous (* $\overline{\dim}_{\mathrm{A}}(\Omega^c) < n$ *), then*

$$\overline{\dim}_{\mathrm{A}}(\Omega^c) < \tfrac{q}{p}(n-p+\beta) \quad \text{or} \quad \underline{\dim}_{\mathrm{M}}(\Omega^c) \geq n-p+\beta\,.$$

In particular, the numbers $\frac{q}{p}(n-p+\beta)$ and $n-p+\beta$ in the sufficient conditions are sharp (although different dimensions in the lower bounds).

Such dichotomy holds also locally: for all balls $B \subset \mathbb{R}^n$ either $\overline{\dim}_{\mathrm{A}}(B \cap \Omega^c) < \frac{q}{p}(n-p+\beta)$ or $\dim_{\mathrm{H}}(2B \cap \Omega^c) \geq n-p+\beta$ when $\beta \geq 0$, and respective bounds hold when $\beta < 0$.

Unweighted characterization

In the results involving a 'thin' complement (corresponding to an upper bound for $\overline{\dim}_{\mathrm{A}}(\Omega^c)$), the HS-inequalities hold actually for all $u \in C_0^\infty(\mathbb{R}^n)$, not only for $u \in C_0^\infty(\Omega)$ as in the 'thick' case. Such inequalities are called *global* Hardy–Sobolev inequalities. In particular, we have the following characterization in the unweighted case $\beta = 0$.

Corollary (LV, 2015)

Let $E \neq \emptyset$ be a closed set in $\mathbb{R}^n$ and let $1 \leq p \leq q < np/(n-p) < \infty$. Then the global $(q,p,0)$-Hardy–Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^q \, \delta_E^{(q/p)(n-p)-n} \, dx\right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx\right)^{1/p}$$

holds for every $u \in C_0^\infty(\mathbb{R}^n)$ if and only if $\overline{\dim}_{\mathrm{A}}(E) < \frac{q}{p}(n-p)$.

Some references:

A. Käenmäki, J. Lehrbäck and M. Vuorinen. Dimensions, Whitney covers, and tubular neighborhoods. *Indiana Univ. Math. J.* 62(6):1861–1889, 2013.

J. Lehrbäck. Weighted Hardy inequalities and the size of the boundary. *Manuscripta Math.*, 127(2):249–273, 2008.

J. Lehrbäck. Hardy inequalities and Assouad dimensions. *J. Anal. Math.* (to appear) arXiv:1402.6134

J. Lehrbäck and A. Vähäkangas. In between the inequalities of Sobolev and Hardy. preprint 2015, arXiv:1502.01190

Some references:

H. AIKAWA. Quasiadditivity of Riesz capacity. *Math. Scand.* 69(1):15–30, 1991.

P. ASSOUAD. Plongements lipschitziens dans $\mathbf{R}^n$.
Bull. Soc. Math. France 111(4):429–448, 1983.

M. BADIALE AND G. TARANTELLO. A Sobolev–Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics.
Arch. Ration. Mech. Anal., 163(4):259–293, 2002.

G. BOULIGAND. Ensembles impropres et nombre dimensionnel.
Bull. Sci. Math. 52:320–344 and 361–376, 1928.

L. CAFFARELLI, R. KOHN, AND L. NIRENBERG. First order interpolation inequalities with weights. *Compositio Math.*, 53(3):259–275, 1984.

J. FRASER. Assouad type dimensions and homogeneity of fractals.
Trans. Amer. Math. Soc. 366(12):6687–6733, 2014.

T. HORIUCHI. The imbedding theorems for weighted Sobolev spaces.
J. Math. Kyoto Univ., 29(3):365–403, 1989.

P. KOSKELA AND X. ZHONG. Hardy's inequality and the boundary size.
Proc. Amer. Math. Soc., 131(4):1151–1158 (electronic), 2003.

J. LUUKKAINEN. Assouad dimension: antifractal metrization, porous sets, and homogeneous measures. *J. Korean Math. Soc.* 35(1):23–76, 1998.

V. G. MAZ'YA. Sobolev spaces. Springer-Verlag, Berlin, 1985.