Quasi - linear PDE's and lower dimensional sets

PDE's, Potential Theory, Function Spaces
Linköping University, June 14-18
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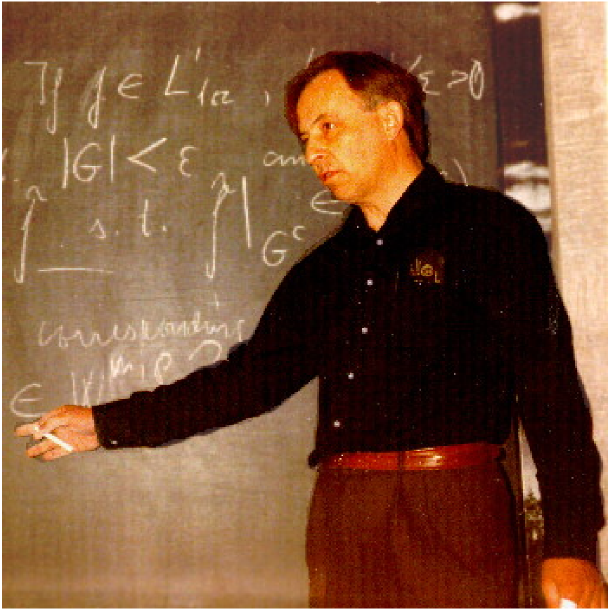
In Memory of Lars Inge Hedberg



Introduction

In this talk we discuss generalizations of a paper with Kaj Nyström and Niklas Lundström in Boundary Harnack Inequalities for Operators of p Laplace Type in Reifenberg Flat Domains, Proceedings of Symposia in Pure Mathematics **79** (2008), 229-266,
regarding boundary Harnack inequalities, the Martin boundary problem, and boundary regularity for non-negative solutions to equations of p-Laplace type vanishing on the boundary of certain sets in Euclidean n space. Our generalization of the above paper (joint with Kaj Nyström) is currently entitled Quasi- linear PDE's and low-dimensional sets. This paper is concerned with the above problems on codimension > 1 sets and for certain values of p. In this case the novelty of our work is that more traditional boundary value problems (eg, boundary value problems for the Laplace operator) require that the boundary have a certain fatness in order that a solution exist.

For example in $\mathbf{R}^3$ the Laplace operator does not see a line segment while if $p > 2$ there are positive solutions (weak) to the p Laplace equation which vanish on the line segment and are p harmonic elsewhere in a neighborhood containing this segment.

Structure Assumptions for Operators of p Laplace Type.

Let $p, \beta, \alpha \in (1, \infty)$ and $\gamma \in (0, 1)$. Let $A = (A_1, ..., A_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, assume that $A = A(y, \eta)$ is continuous in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and that $A(y, \eta)$, for fixed $y \in \mathbb{R}^n$, is continuously differentiable in η_k, for every $k \in \{1, ..., n\}$, whenever $\eta \in \mathbb{R}^n \setminus \{0\}$. Assume that the following conditions are satisfied whenever y, x, $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n \setminus \{0\}$:

$$
\begin{aligned}
&(i) \qquad \alpha^{-1}|\eta|^{p-2}|\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial A_i}{\partial \eta_j}(y, \eta)\xi_i\xi_j \leq \alpha|\eta|^{p-2}|\xi|^2, \\
&(ii) \qquad |A(x, \eta) - A(y, \eta)| \leq \beta|x - y|^{\gamma}|\eta|^{p-1}, \\
&(iii) \qquad A(y, \eta) = |\eta|^{p-1}A(y, \eta/|\eta|).
\end{aligned}
\tag{1}
$$

We say that u is A-harmonic or a weak solution to $\nabla \cdot A(x, \nabla u) = 0$ in a bounded open set G provided $u \in W^{1,p}(G)$ and whenever $\theta \in W_0^{1,p}(G)$

$$\int \langle A(y, \nabla u(y)), \nabla \theta(y) \rangle \, dy = 0, \tag{2}$$

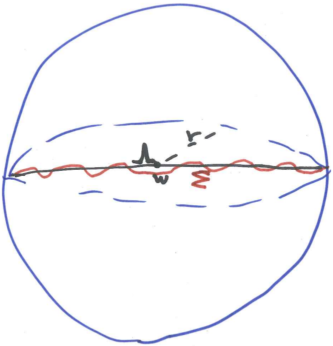
Note that if $A(x, \eta) = |\eta|^{p-2}\eta$, then u is said to be p-harmonic in G.

Definition 1. Let n, m, be integers such that $1 \leq m \leq n-1$. Let $\Sigma \subset \mathbb{R}^n$ be a closed set and let $r_0, \delta > 0$ be given. We say that Σ is (m, r_0, δ)-Reifenberg flat (in $\mathbb{R}^n$) if there exists, whenever $w \in \Sigma$ and $0 < r < r_0$, an m dimensional hyperplane $\Lambda = \Lambda_m(w, r)$ such that

$$h(\Sigma \cap B(w, r), \Lambda \cap B(w, r)) \leq \delta r.$$

In this display $h(\cdot, \cdot)$ denotes Hausdorff distance between the two sets. In general if $E, F \subset \mathbf{R}^n$ then the Hausdorff distance between E, F is,

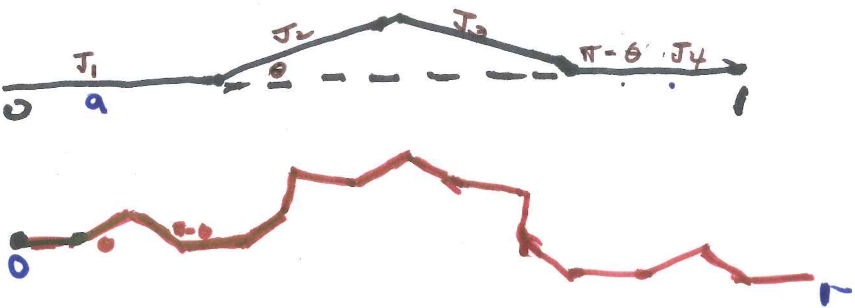
$$h(E, F) = \max(\sup\{d(y, E) : y \in F\}, \sup\{d(y, F) : y \in E\}).$$



Definition 2. Σ is said to be (m, r_0, δ)-
Reifenberg flat with vanishing constant if in addition, for each $\epsilon > 0$, there exists $\tilde{r} = \tilde{r}(\epsilon) > 0$ such that whenever $x \in \Sigma \cap B(w, r)$ and $0 < \rho < \tilde{r}$, there is an m dimensional hyperplane $\Lambda' = \Lambda'_m(x, \rho)$ through x with

$$h(\Sigma \cap B(x, \rho), \Lambda' \cap B(x, \rho)) \leq \epsilon\rho.$$

Remark. m Reifenberg flat sets Σ are well approximated on all small scales by m dimensional planes. Still Σ may not have an m dimensional tangent plane (in the classical sense) at any point in it, no matter how small δ is. For example given $\theta, 0 < \theta < \pi/2$, let $a^{-1} = 2 + 2\cos\theta$. Let $J_1 = [0, a]$. Let J_2 be the line segment with endpoints $(a, 0)$ and $(a + a\cos\theta, a\sin\theta)$. Let J_3 be the line segment with endpoints, $(a + a\cos\theta, a\sin\theta)$ and $(1 - a, 0)$. Let $J_4 = [1 - a, 1]$. Note that all four intervals have sidelength a. Also J_2, J_3 make angles $\theta, \pi - \theta$ with (0, 1). On each of these line segments we can now repeat the process (i.e replace each segment by four equal segments with the first and last segments contained in the given segment and with the second, third segments making angles $\theta, \pi - \theta$ with the given segment). Continuing in this way we get a $(1, 1, \delta)$ Reifenberg flat set provided θ is small enough. Moreover this set does not have a tangent line at any point in it.



We require more assumptions on A for our theorems when $2 \leq m \leq n-2$ and $n \geq 4$. Thus we first consider the case when $m = 1$.

Theorem A

Fix $p, n-1 < p < \infty$, and $n \geq 3$. Let $\Sigma \subset \mathbb{R}^n$ be a closed set and assume that Σ is $(1, r_0, \delta)$-Reifenberg flat (in $\mathbb{R}^n$) for some $r_0, \delta > 0$. Let $w \in \Sigma, 0 < r < r_0$. Assume that u, v are positive A-harmonic functions in $B(w, 4r) \setminus \Sigma$, continuous on $B(w, 4r)$ and $u = 0 = v$ on $\Sigma \cap B(w, 4r)$. Then there exist $\bar{\delta} = \bar{\delta}(p, n, \alpha, \beta, \gamma) > 0$, $\bar{c} = \bar{c}(p, n, \alpha, \beta, \gamma) \geq 1$ and $\bar{\sigma} = \bar{\sigma}(p, n, \alpha, \beta, \gamma) > 0$, such that if $0 < \delta < \bar{\delta}$, then

$$\left| \log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)} \right| \leq c \left(\frac{|y_1 - y_2|}{r} \right)^{\sigma}$$

whenever $y_1, y_2 \in B(w, r/c) \setminus \Sigma$.

To state a theorem similar to Theorem A when $2 \le m \le n-2$, we suppose A in addition to the structure conditions listed in (1) satisfies either (3) (a) or (3) (b).

(a) There exists $0 < \lambda < \infty$ with $|\frac{\partial A_i}{\partial \eta_j}(x,\eta) - \frac{\partial A_i}{\partial \eta_j'}(x,\eta')| \le \lambda\,|\eta - \eta'||\eta|^{p-3}$ for $x \in \mathbf{R}^n, 1 \le i,j \le n$ and $\eta, \eta' \in \mathbf{R}^n \setminus \{0\}$ with $\frac{1}{2}|\eta| \le |\eta'| \le 2|\eta|$.

(b) $A(x,\eta) = \kappa(x,\eta)\,|\langle C(x)\eta, \eta\rangle|^{p-2}\,C(x)\eta,\ x \in \mathbf{R}^n, \eta \in \mathbf{R}^n \setminus \{0\},$ where $C(x)$ is a linear transformation of $\mathbf{R}^n$ and $\kappa(x,\cdot)$, is homogeneous of degree 0 in η whenever $x \in \mathbf{R}^n$.

(3)

Theorem B

Let $2 \leq m \leq n-2, n \geq 4$, and $n-m < p < \infty$, be given. Let $\Sigma \subset \mathbb{R}^n$ be a closed (m, r_0, δ)-Reifenberg flat set (in $\mathbb{R}^n$) for some $r_0, \delta > 0$. Let w, r, r_0, u, v be defined as in Theorem A relative to Σ. Suppose also that in addition to (1) $(i)-(iii)$ that A satisfies either $3(a)$ or (b). Then the conclusion of Theorem A is valid with $\bar{\delta}, \bar{c}, \bar{\sigma}$ replaced by δ', c', σ'. Constants now may also depend on m, λ.

Remark. Theorems A, B, imply u/v is bounded and σ Hölder continuous in $B(w, r/c) \setminus \Sigma$. That is, for some c',

$$\left| \frac{u(y)}{v(y)} - \frac{u(z)}{v(z)} \right| \leq c' \left(\frac{|z-y|}{r} \right)^{\sigma} \frac{u(z)}{v(z)} \qquad (4)$$

whenever $y, z \in B(w, r/c')$ and $0 < r \leq r_0$. Thus the conclusions in Theorems A, B, are usually referred to as ‘ boundary Harnack inequalities.’

In order to state corollaries to Theorems A, B we remark that if u is as in these theorems, then there exists a positive Borel measure μ on $B(w,4r)\cap\Sigma$ with

$$\int \langle A(x,\nabla u),\nabla\phi\rangle\, dx = -\int \phi d\mu \tag{5}$$

whenever $\phi\in C_0^\infty(B(w,4r))$. Using Theorems A, B, we prove

Corollary A Let $u, v, m, p, \sigma, \Sigma$, be as in Theorems A, B, and μ,ν, the corresponding measures as in (5). Then there exists f in $C^\sigma(\Sigma\cap\bar{B}(w,r))$ with $d\mu = f\,d\nu$.

Corollary B Let n, m, p, Σ, r_0, A, w, u, μ, be as in Corollary A and suppose that in addition, $\Sigma\cap B(w,4r_0)$ is m Reifenberg flat with vanishing constant. Then

$$\lim_{r\to 0}\frac{\mu(B(x,tr))}{\mu(B(x,r))} = t^m \text{ uniformly for } x\in\Sigma\cap\bar{B}(w,r_0) \text{ and } t\in[1/2,1].$$

In the language of T. Toro and coauthors, a measure μ is said to be asymptotically optimally doubling on $\Sigma \cap \bar{B}(w, r_0)$ if the conclusion of Corollary B holds for μ. For optimal doubling when $m = n - 1$ see Avelin, Lundström, and Nyström, Optimal doubling, Reifenberg fllatness and operators of p-Laplace type, Nonlinear Anal. 74 (2011), no. 17, 5943-5955.

Theorem C

Let n, m, be integers such that $1 \leq m \leq n - 2$ and let p, $n - m < p < \infty$, be given. Let $\Sigma \subset \mathbb{R}^n$ be a closed set and assume that Σ is (m, r_0, δ)-Reifenberg flat (in $\mathbb{R}^n$) for some $r_0, \delta > 0$. Then there exists $\delta^* = \delta^*(p, n, \alpha, \beta, \gamma)$ such that the following is true whenever $0 < \delta < \delta^*$, $w \in \Sigma, 0 < r < r_0$. Suppose that $\hat{u}, \hat{v}$ are positive A-harmonic functions in $B(w, 4r) \setminus \Sigma$, continuous on $\bar{B}(w, 4r) \setminus \{w\}$ and $\hat{u} = 0 = \hat{v}$ on $\partial(B(w, 4r) \setminus \Sigma) \setminus \{w\}$. For $1 < m \leq n - 2$ assume also that either 3(a) or (b) hold. If $0 < \delta < \delta^*$, then $\hat{u}(y) = \chi \hat{v}(y)$ for all $y \in B(w, 4r) \setminus \Sigma$ and for some constant χ.

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Remark. Theorems A, B and C for $m = n - 1$ were proved by Lundström, Nyström, and myself in the paper mentioned earlier. Theorem C implies that the A Martin boundary agrees with the topological boundary of Σ

see R.S. Martin, Minimal positive harmonic functions, Trans. Amer. Math. Soc **49** (1941), 137-172.

Outline of the Proof of Theorems A, B

In the proof of Theorems A, B, we in general follow the proof scheme in the paper of Lewis, Lundström, and Nyström, mentioned earlier. However proofs are more involved and often involve considerable expertise. In fact for a general A as in (1), our argument breaks down in one key place when $1 < m \leq n - 2$. To begin the proof of Theorems A, B, we note that the (m, r_0, δ) Reifenberg flat assumption in Theorems A, B and the assumptions on p imply that Σ is uniformly fat in the sense of p capacity (for p capacities see Adams and Hedberg, Function spaces and Potential Theory, Springer **314**, 1996)

Using uniform p fatness we first prove

Lemma A Let $r_0, \delta, m, n, p, \Sigma, w, r$, and u be as in Theorems A, B. Then

$$(i) \qquad r^{p-n} \int_{B(w,r/2)} |\nabla u|^p \, dy \le c \, (\max_{B(w,r)} u)^p.$$

Furthermore, there exists $\sigma = \sigma(p, n, m, \alpha, \beta, \gamma) \in (0, 1)$ such that if $x, y \in B(w, r)$, then

$$(ii) \qquad |u(x) - u(y)| \le c \left(\frac{|x-y|}{r} \right)^{\sigma} \max_{B(w,2r)} u.$$

Remark. The first inequality is a standard Caccioppoli inequality while the last inequality follows from uniform p fatness of Σ together with Wiener estimates for A harmonic functions as in Heinonen, Kilpelainen, Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Dover Publications, 2006),

.

Lemma B Let u, Σ, w, r and all other data be as in Lemma A. Then there exists $c = c(p, n, m, \alpha, \beta, \gamma)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then

$$\max_{B(w,\tilde{r})} u \leq c\, u(a_{\tilde{r}}(w))$$

where $a_{\tilde{r}}(w)$ is a point in $B(w, \tilde{r})$ whose distance to Σ is $\approx \tilde{r}$.

Lemma C Let u, Σ, w, r and all other data be as in Lemma B. Then there exists $\hat{\sigma} \in (0, 1]$, depending only on p, n, m, α, β, γ, such that if $x, y \in B(\hat{w}, \hat{r}/2)$, $B(\hat{w}, 4\hat{r}) \subset B(w, 4r) \setminus \Sigma$, then

$$\begin{aligned} c^{-1}\, |\nabla u(x) - \nabla u(y)| &\leq (|x-y|/\hat{r})^{\hat{\sigma}} \max_{B(\hat{w},\hat{r})} |\nabla u| \\ &\leq c\, \hat{r}^{-1}\, (|x-y|/\hat{r})^{\hat{\sigma}} \min_{B(\hat{w},2\hat{r})} u. \end{aligned}$$

Lemma D Let $u, \Sigma, w, r, \tilde{r}$ and all other data be as in Lemma B. If μ is the measure corresponding to u as in (5), then

$$\tilde{r}^{p-n} \mu(B(w, \tilde{r})) \approx u(a_{\tilde{r}}(w))^{p-1}.$$

Lemma E

Let $O \subset \mathbb{R}^n$ be an open set, $1 < p < \infty$, and A_1, A_2 be as in (1) . Let $\hat{u}_1$ be A_1-harmonic and let $\hat{u}_2$ be A_2-harmonic in O. Let $\tilde{a} \geq 1, y \in O$ and suppose that

$$\frac{1}{\tilde{a}} \frac{\hat{u}_1(y)}{d(y, \partial O)} \leq |\nabla \hat{u}_1(y)| \leq \tilde{a} \frac{\hat{u}_1(y)}{d(y, \partial O)}.$$

Let $\tilde{\epsilon}^{-1} = (c\tilde{a})^{(1+\hat{\sigma})/\hat{\sigma}}$, where $\hat{\sigma}$ is as in Lemma C. If

$$(1 - \tilde{\epsilon})\hat{L} \leq \frac{\hat{u}_2}{\hat{u}_1} \leq (1 + \tilde{\epsilon})\hat{L} \text{ in } B(y, \tfrac{1}{100} d(y, \partial O))$$

for some $\hat{L}, 0 < \hat{L} < \infty$, then for $c = c(p, n, \alpha, \beta, \gamma)$ suitably large,

$$\frac{1}{c\tilde{a}} \frac{\hat{u}_2(y)}{d(y, \partial O)} \leq |\nabla \hat{u}_2(y)| \leq c\tilde{a} \frac{\hat{u}_2(y)}{d(y, \partial O)}.$$

.

Lemmas $B - E$ are essentially copied verbatim from earlier work of Lewis and Nystrom. Lemma E whose proof is elementary given Lemmas A- D is of fundamental importance in the proof of Theorems A, B. Armed with Lemmas $A - E$ we proceed by the following steps:

Step 1 Let u, v, p, w, r and all other quantities be as in Theorems A or B. Let

$$\Sigma = \mathbf{R}^m \times \{(0, \ldots, 0)\} = \{x = (x', x'') : x'' = (0, \ldots, 0)\}.$$

Then there exists $c \geq 1$ (depending only on the data) such that if $u(a_{\tilde{r}}(w)) \approx v(a_{\tilde{r}}(w)) \approx 1$,

$$(6) \quad c^{-1} \leq u(x)/v(x) \leq c \text{ whenever } x \in B(w, r/c) \setminus \Sigma.$$

(6) is the inequality we have not been able to prove for a general A as in (2) when $2 \leq m \leq n - 2$. For $m = n - 1$ we were able to prove this estimate in the paper mentioned earlier, using more or less standard barriers, i.e by constructing a subsolution or supersolution to (2) which lies below or above a given solution and has the desired properties. For $m = 1$ we use a technique of Bennewitz and Lewis in

.

On the Dimension of p Harmonic Measure, Ann. Acad. Sci. Fenn. **30** (2005), 459-505.

Unfortunately this technique only works for $m = 1$ or if the PDE is rotationally invariant.

To handle the case when $2 \leq m \leq n - 2$. we use (3) to construct a not so standard barrier. To make the construction let $A = A(\eta)$ be as in (1) and set $\tilde{A}_j = A_{m+j}, 1 \leq j \leq n - m$. Given $p > n - m$ let $\tilde{u}$ be the 'fundamental solution' in $\mathbf{R}^{n-m}$ corresponding to $\tilde{A}$ with pole at $x = 0$. That is, $\tilde{u}$ is continuous on $\mathbf{R}^{n-m}$ with $\tilde{u}(0) = 0$ and locally in $W^{1,p}(\mathbf{R}^{n-m})$. Moreover

$$\int_{\mathbf{R}^{n-m}} \langle \tilde{A}(\nabla \tilde{u}), \nabla \phi \rangle \, dx \; = \; -\phi(0) \tag{7}$$

whenever $\phi \in C_0^\infty(\mathbf{R}^{n-m})$. Existence and uniqueness of $\tilde{u}$ satisfying (7) are not so difficult to prove. We show that

$$\tilde{u}(z) = |z|^{\xi}\, \tilde{u}(\tfrac{z}{|z|}) \text{ and } |\nabla \tilde{u}|(z) \approx \tilde{u}(z)/|z|, z \in \mathbf{R}^{n-m} \setminus \{(0, \ldots, 0)\} \tag{8}$$

where $\xi = \frac{p-n+m}{p-1}$. Let $\hat{u}(x', x'') = \tilde{u}(x'')$ when $x = (x', x'') \in \mathbf{R}^n$.
Remark. Note that $\hat{u} \equiv 0$ on $\mathbf{R}^m \times \{(0, \dots, 0)\}$ so can be used as a comparison function in (6).

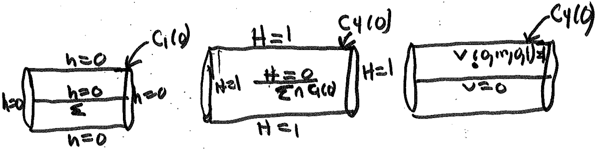
To outline the proof of (6) when $2 \leq m \leq n-2, r = 1$, and $w = 0$, let $C_\rho(0) = \{x = (x', x'') : |x'| \leq \rho, |x''| \leq \rho\}$.

$$h(x) = (e^{\hat{u}(x)} - 1)(1 - |x'|^2) \text{ when } x \in C_1(0)$$

Then under either assumption (3) (a) or (b) we show in our new paper that

$$h(x) \leq c\, v(x) \text{ whenever } x \in C_{1/2}(0). \tag{9}$$

Here v is A harmonic in $C_4(0)$ with $v(0, \dots, 0, 1) = 1$ and continuous boundary value 0 on $(\mathbf{R}^m \times \{(0, \dots, 0)\}) \cap C_4(0)$. To finish the proof of (6) in this situation let H be the A harmonic function in $C_4(0) \setminus \{(x', 0, \dots, 0) : |x'| \leq 1\}$ with continuous boundary values $H(x', 0) \equiv 0$ when $|x'| \leq 1$ and $H \equiv 1$ on $\partial C_4(0)$. From the maximum principle for A harmonic functions we find for $v \leq c'$ as above that $v \leq c' H$ in $C_1(0)$. In view of (8), (9), we see that



in order to finish the proof of (6) it suffices to show for some c depending only on the data that

$$H(x) \approx h(x) \approx \hat{u}(x) \approx |x''|^{\xi} \text{ for } x \in C_{1/c}(0) \setminus \Sigma. \tag{10}$$

To do this we begin by showing that H satisfies the fundamental inequality :

$$c^{-1} d(x,\Sigma)^{-1} H(x) \leq |\nabla H(x)| \leq c d(x,\Sigma)^{-1} H(x) \tag{11}$$

when $x \in C_2(0) \setminus (\mathbf{R}^m \times \{(0,\ldots,0)\}$ where $c \geq 1$ depends only on the data. Second we proceed to step 2.

Step 2: In this step we finish the proof of Theorems A, B when Σ is the m dimensional plane in Step 1. To outline this step we first prove

Lemma F

Let Σ be the m dimensional plane in Step 1, $w \in \Sigma$ and let u_1, u_2, be A harmonic in $B(w, 4r) \setminus \Sigma$ for fixed p as in Theorem 1 with continuous boundary values, $u_1 \equiv 0 \equiv u_2$ on $\Sigma \cap B(w, 4r)$. Suppose also for some $c_1 \geq 1$ and $x \in B(w, r/c_1)$ that

$$c_1^{-1} u_i(x)/d(x,\Sigma) \leq |\nabla u_i|(x) \leq c_1 u_i(x)/d(x,\Sigma), \tag{12}$$

when $i = 1, 2$. There exists c_2 depending on c_1 and the other data in Theorem 1 but independent of u_1, u_2 such that if $a, b \in [0, \infty)$, then $(a|\nabla u_1| + b|\nabla u_2|)^{p-2}$ is an A_2 weight with constant $\leq c_2$ on cubes contained in $B(w, r/c_2)$.

Lemma G Let u_1, u_2 be as in Lemma F. Then Theorem A is valid with u, v replaced by u_1, u_2 when Σ is an m dimensional plane.

Using Lemma F we get Lemma G by using boundary Harnack inequalities for solutions to degenerate elliptic PDE in divergence form whose degeneracy is given in terms of an A_2 weight.

These inequalities are derived in

Fabes, C. Kenig & R. Serapioni, The local regularity of solutions to degenerate elliptic equations, Comm. Partial Differential Equations **7** (1982), 77–116.

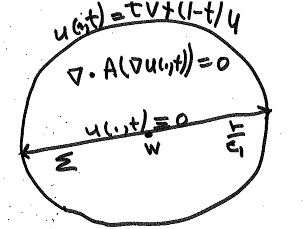
E. Fabes, D. Jerison , C. Kenig, The Wiener test for degenerate elliptic equations, Ann. Inst. Fourier (Grenoble) **32** (1982), 151–182.

E. Fabes, D. Jerison, C. Kenig, Boundary behaviour of solutions to degenerate elliptic equations, Conference on harmonic analysis in honor of Antonio Zygmund, Vol I, II Chicago, Ill, 1981, 577-589, Wadsworth Math. Ser, Wadsworth Belmont CA, 1983.

From (8), (11), we deduce that Lemmas F, G can be applied with $H = u_1, \hat{u} = u_2, w = 0$, and $r = 1$. Doing this we get (10) and so also (6) in $C_{1/c}(0)$ provided $c \geq 1$ is large enough depending only on the data.

Finally in step 2 we prove Theorems A, B for u, v when Σ is the m dimensional plane in Step 1 and u satisfies

(12) in $B(w, r/c_1) \setminus \Sigma$ with u_i replaced by u. To do this let $u(\cdot, t)$ be the A harmonic function in $B(w, r/c_1) \setminus \Sigma$ with continuous boundary values $tv + (1-t)u$.



We assume as we may that $v(a_{\tilde{r}}(w)) \approx u(a_{\tilde{r}}(w))$. From Lemmas B-E and the boundary Harnack inequality in (6) it follows that if ϵ is small enough and $t \in [1-\epsilon, 1]$ then $u(\cdot, t)$ satisfies (12) in $B(w, r/c_1)$ with constants as in Lemma E. For $t \in [1-\epsilon, 1]$ we can then apply Lemmas F and G to get that the conclusion of Theorems A, B hold with v replaced by $u(\cdot, t)$. From Theorem A it follows easily that $\frac{u(\cdot,t)}{u}$ is Hölder continuous in $B(w, \tilde{r}) \setminus \Sigma$. Hölder continuity of this ratio leads to an induction type argument where we alternatively use Lemmas E,F, G, and then Theorems A, B in intervals $[t_m, t_{m+1}]$ with $t_{m+1} - t_m \geq \epsilon_1$ for some $\epsilon_1 > 0$ to get first that $u(\cdot, t_m)$ satisfies (12) with constants independent of m and thereupon that $u(\cdot, t)/u$ is Hölder continuous for $t \in [t_m, t_{m+1}]$. Now ϵ_1 can be chosen independently of m so eventually we get from Lemma E that $|\nabla v(x)| \approx v(x)/d(x, \Sigma)$ and thereupon that Theorems A, B hold for u, v. and Σ an m dimensional plane.

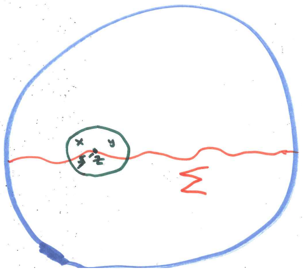
Step 3 In this step we prove Theorems A, B in the generality stated. In this case we use Theorems A, B when Σ is an m plane and Lemma E to show that u, v satisfy the fundamental inequality (12). We also show for given $a, b > 0$ and some $c' \geq 1$ that $(a|\nabla u| + b|\nabla v|)^{p-2}$ is an A_2 weight on sub cubes of $B(w, r/c')$ with constants which are independent of a, b. Armed with these facts, we can conclude Theorems A, B from the boundary Harnack Inequalities of Fabes et al, mentioned earlier.

Outline of the Proof of Corollary A

Let $u, v, n, m, p, \Sigma, w, r_0, A, \sigma$ be as in Theorems A, B, and let $\mu,\ \nu,$ be the corresponding measures as in (5). If $z \in B(w, 2r_0) \setminus \Sigma$. Recall from (4) with w replaced by z that Theorems A, B imply

$$\left| \frac{u(x)}{v(x)} - \frac{u(y)}{v(y)} \right| \leq c \frac{u(x)}{v(x)} \left(\frac{|x-y|}{r} \right)^{\sigma} \tag{13}$$

whenever $x, y \in B(z, r) \setminus \Sigma$ and $0 < r \leq r_0/c$. From (13) we deduce



that $0 < f(z) = \lim\limits_{y \to z} \dfrac{u(y)}{v(y)}$ exists and that (13) holds with $\frac{u(y)}{v(y)}$ replaced by $f(z)$. So there exists c' depending only on the data such that if $0 < s < r$ and $x \in B(z,s) \setminus \Sigma$, then

$$u(x)(1 - c'(s/r)^\sigma) < f(z)\, v(x) < u(x)(1 + c'(s/r)^\sigma). \qquad (14)$$

Set

$$\tau_1 = \frac{f(z)}{(1 + c'(s/r)^\sigma)}, \quad \tilde{v}(x) = \tau_1 v, \text{ and } h = u - \tilde{v} > 0 \text{ in } B(z,s) \setminus \Sigma.$$

If $\psi \in C_0^\infty(B(z,s))$ and θ_1, θ_2, are small positive numbers we put $\phi = \max\{h - \theta_1, 0\}^{\theta_2}\psi$. Using (1) we see that

$$0 \leq \int \langle A(x, \nabla u) - A(x, \nabla \tilde{v}), \nabla(\max\{h - \theta_1, 0\}^{\theta_2})\rangle \psi \, dx$$

Also from the usual limiting argument we find that ϕ can be used as a test function in the weak formulation of A harmonicity for both $u, \tilde{v}$. Doing this, using (2), (5), and letting first $\theta_1 \to 0$, and then $\theta_2 \to 0$, we conclude that

$$\int \psi \, (\tau_1^{p-1} d\nu - d\mu) \leq \int_{B(z,s)} \langle A(x, \nabla u) - A(x, \nabla \tilde{v}), \nabla \psi \rangle \, dx \ \leq 0,$$

where we have also used $p-1$ homogeneity of A in (1) (iii) to deduce the measure corresponding to $\tilde{v}$. From arbitrariness of ψ it follows that $\tau_1^{p-1}\nu \leq \mu$ on $B(z,s)\cap\Sigma$. Similarly if $\tau_2 = \frac{f(z)}{(1-c'(s/r)^\sigma)}$ then $\mu \leq \tau_2^{p-1}\nu$. on $B(z,s)\cap\Sigma$. From this discussion we see that μ,ν are mutually absolutely continuous on $B(w,4r_0)$ and if $d\mu = k\,d\nu$, then

$$\tau_1^{p-1} \leq k(\hat{z}) \leq \tau_2^{p-1} \text{ when } \hat{z} \in B(z,s)\cap\Sigma \text{ and } k(z) = f(z)^{p-1}.$$

Taking logarithms it follows that

$$c^{-1}(s/r)^\sigma \;\leq\; |\log(k(\hat{z})/k(z))| \;\leq c(s/r)^\sigma$$

for some $c \geq 1$ depending only on the data. From this display and arbitrariness of s, z we conclude that Corollary A is valid.

Proof of Corollary B

The proof of Corollary B is by contradiction. If this corollary is false there exists $t_j \in [1/2, 1], x_j \in \Sigma \cap \bar{B}(w, r_0), 0 < r_j \leq 10^{-j} r_0$, for $j = 1, \dots,$ and $\epsilon > 0$ for which

$$\epsilon \leq \left| \frac{\mu(B(x_j, t_j r_j))}{\mu(B(x_j, r_j))} - t_j^m \right|. \tag{15}$$

We assume as we may that $x_j \to \hat{x} \in \Sigma \cap \bar{B}(w, r_0)$ as $j \to \infty$ and $t_j \to t \in [1/2, 1]$ as $j \to \infty$. Let

$$u_j(x) = \frac{u(x_j + r_j x)}{u(a_{r_j}(x_j))} \text{ when } x \in \Omega_j = \{x : x_j + r_j x \in B(w, 2r_0) \setminus \Sigma\}.$$

Let $A_j(x, \eta) = A(x_j + r_j x, \eta)$ when $x, \eta \in \mathbf{R}^n$. From $p - 1$ homogeneity of A in (1) (*iii*) we see that u_j is a weak solution to $\nabla \cdot A_j(x, \nabla u_j) = 0$ in Ω_j. Note that A_j has the same structure constants as A in (*i*), (*iii*), of (1). while β in (*ii*) is replaced by βr_j^γ.

From the vanishing Reifenberg flat assumption in Corollary B we see that for a subsequence of (Ω_j) (also denoted (Ω_j)), we have $\partial\Omega_j \to \Lambda$, an m dimensional plane through 0, as $j \to \infty$, uniformly in the Hausdorff distance sense on compact subsets of $\mathbf{R}^n$. From Lemmas A, B, as well as Harnack's inequality, and the NTA property of Ω_j we see that given $R > 0$, there exists j_0 such that u_j is Hölder continuous with exponent σ and uniformly bounded Hölder norm in $B(0, R)$ when $j \geq j_0$. Also given K, a compact subset of $\mathbf{R}^n \setminus \Lambda$, we find from Lemma C that ∇u_j is $\hat{\sigma}$ Hölder continuous on K with a uniformly bounded Hölder norm for j large enough. Furthermore, from these lemmas, we conclude that (u_j) is bounded in the norm of $W^{1,p}(B(0, R))$.

Using these facts we obtain from Ascoli's theorem that subsequences of $(u_j), (\nabla u_j)$ (also denoted $(u_j), (\nabla u_j)$), converge uniformly on compact subsets of $\mathbf{R}^n, \mathbf{R}^n \setminus \Lambda$, to $\bar{u}, \nabla \bar{u}$. From weak compactness of $W^{1,p}$ we may also assume that $u_j \to \bar{u}$ weakly in $W^{1,p}(B(0, R))$ for each $R > 0$. Then $\bar{u}$ is σ Hölder continuous in $\mathbf{R}^n$ and $\bar{u} \equiv 0$ on Λ. Also it is easily seen that $\bar{u}$ is $\hat{A}$ harmonic in $\mathbf{R}^n \setminus \Lambda$ with $\hat{A}(\eta) = A(\hat{x}, \eta)$, $\eta \in \mathbf{R}^n$. To reach a contradiction we assume, as we may, that $\Lambda = \mathbf{R}^m \times \{0\}$, We claim that $\bar{u}$ is a constant multiple of $\hat{u}$ defined after (8). This claim is proved by first applying Theorems A or B with u, v replaced by $\bar{u}, \hat{u}$, and then letting $r \to \infty$. Using our claim and Lemma D we deduce that the measure, say $\bar{\mu}$, corresponding to $\bar{u}$, is a constant multiple of Lebesgue measure on $\mathbf{R}^m \times \{0, \dots, 0\}$. Let μ_j be the measure corresponding to u_j, for $j = 1, 2, \dots$ Using the above convergence results, we easily deduce that $\mu_j \to \bar{\mu}$ weakly as measures. Thus

$$\limsup_{j\to\infty} \frac{\mu_j(B(0,t_j))}{\mu_j(B(0,1))} \leq \frac{\hat{\mu}(\bar{B}(0,t))}{\hat{\mu}(B(0,1))} = t^m \leq \liminf_{j\to\infty} \frac{\mu_j(B(0,t_j))}{\mu_j(B(0,1))}. \tag{16}$$

Finally we note from $p-1$ homogeneity of A that

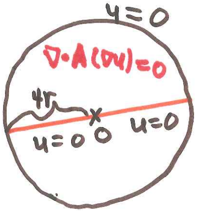
$$\frac{\mu_j(B(0,t_j))}{\mu_j(B(0,1))} = \frac{\mu(B(x_j,t_j r_j))}{\mu_j(B(x_j,r_j))} \text{ for } j = 1, 2, \dots \tag{17}$$

Using (15)-(17) we deduce that

$$\epsilon \leq \lim_{j\to\infty} \left| \frac{\mu(B(x_j,t_j r_j))}{\mu(B(x_j,r_j))} - t_j^m \right| = \lim_{j\to\infty} \left| \frac{\mu_j(B(0,t_j))}{\mu_j(B(0,1))} - t_j^m \right| = 0. \tag{18}$$

We have reached a contradiction. Hence Corollary B is valid.

The proof of Theorem C is similar to the proof of Theorems A, B, in that we first prove this theorem when $\Sigma = \mathbf{R}^m \times \{(0, \dots, 0)\}, A = A(\eta)$, and $w = 0$. To do this we first construct an A Martin function u in $B(0, 4r) \setminus \Sigma$ satisfying

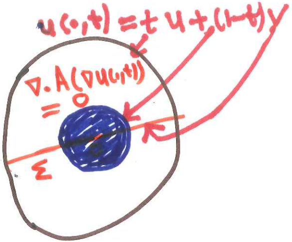


$$|\nabla u(x)| \approx u(x)/d(x, \Sigma), \text{ when } x \in B(0, r) \setminus \Sigma. \tag{19}$$

Let v be another Martin function relative to 0 and $B(0, 4r) \setminus [\mathbf{R}^m \times \{(0, \dots, 0)\}]$. From Theorems A, B, and the maximum principle for A harmonic functions it is easily shown that

$$u(x)/v(x) \approx u(a_r(0))/v(a_r(0)) \tag{20}$$

Also using (19) and Theorems A, B, we can essentially repeat our argument in step 2 involving $u(\cdot, t)$ only now $u(\cdot, t), t \in (0, 1)$ is A harmonic in $B(0, 4r) \setminus [B(0, s) \setminus \Sigma]$ with continuous boundary values = $tu + (1 - t)v$.



in order to obtain first that v also satisfies (19) with constants depending only on the data in $B(0, r) \setminus \Sigma$. Arguing as in the proof of Lemmas F and G, we then obtain for some $\hat{c} \geq 1, \hat{a} \in (0, 1)$, depending only on the data,that

$$\text{osc}\,(t) \leq \hat{c} \left(\frac{s}{t}\right)^{\hat{a}} \text{osc}\,(s), \; s \leq t \leq r. \tag{21}$$

where

$$m(t) = \inf_{\partial B(0,t)} \frac{u}{v}, \; M(t) = \sup_{\partial B(0,t)} \frac{u}{v}, \; \text{osc}\,(t) = M(t) - m(t),$$

Theorem C follows from (21) for $A \in M_p(\alpha)$, if we let $s \to 0$. The general case of Theorem C follows from the above baseline case, in a way similar to the proof of the general case of Theorems A, B. In fact for $2 \leq m \leq n-2$, one can also use a blowup type argument as in the proof of Corollary B to get Theorem C (thanks to 3 (a), (b)).

Food For Thought

Let u, Σ be as in Theorems A or B. We would like to know what extra conditions on Σ imply that the measure μ corresponding to u as in (5) is absolutely continuous with resepect to m dimensional Hausdorff measure on Σ. For example suppose $m = 1, n = 3$, while $\Sigma = \{(x, y, z) : z = 0 \text{ and } y = \phi(x), -\infty < x < \infty\}$. If ϕ is Lipschitz with compact support is it true that $d\mu = f dH^m$ on Σ where f is integrable with respect to H^m measure.

If in addition, ϕ' is Hölder continuous, is it true that $d\mu = f\, dH^m$ where f is Hölder continuous. If $m = n - 1$ the answer to both these questions is yes.

Thanks for Listening!!!