Comparison of Navier and Dirichlet fractional Laplacians

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1) Communications in PDEs, V. 39. 2014. N3

2) Preprint http://arxiv.org/abs/1408.3568

3) Preprint http://arxiv.org/abs/1503.00271

$\Omega \Subset \mathbb{R}^n$ is a smooth domain.

First let us consider *polyharmonic* operators. The Navier BC for $(-\Delta)^k$, $k \in \mathbb{N}$, are defined as follows:

$$u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = \Delta^2 u\big|_{\partial\Omega} = \cdots = \Delta^{k-1} u\big|_{\partial\Omega} = 0.$$

The corresponding operator $(-\Delta_\Omega)^k_N$ can be defined by its quadratic form

$$((-\Delta_\Omega)^k_N u, u) = \sum_j \lambda_j^k \cdot |(u, \varphi_j)|^2.$$

Here, λ_j and φ_j are eigenvalues and eigenfunctions of the Dirichlet Laplacian in Ω, respectively.

The Dirichlet BC for the operator $(-\Delta)^k$ are

$$u\Big|_{\partial\Omega} = \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = \frac{\partial^2 u}{\partial \mathbf{n}^2}\Big|_{\partial\Omega} = \cdots = \frac{\partial^{k-1} u}{\partial \mathbf{n}^{k-1}}\Big|_{\partial\Omega} = 0,$$

where $\mathbf{n}$ is the unit exterior normal vector to $\partial\Omega$. The quadratic form of corresponding operator $(-\Delta_\Omega)^k_D$ is the restriction of the quadratic form for $(-\Delta)^k$ in $\mathbb{R}^n$ to functions supported in Ω:

$$((-\Delta_\Omega)^k_D u, u) = \int_{\mathbb{R}^n} |\xi|^{2k} |\mathcal{F}u(\xi)|^2 d\xi,$$

where $\mathcal{F}$ is the Fourier transform.

Now for arbitrary $s > -1$ we define the "Navier" fractional Laplacian by the quadratic form

$$Q^N_{s,\Omega}[u] = ((-\Delta_\Omega)^s_N u, u) := \sum_j \lambda_j^s \cdot |(u, \varphi_j)|^2$$

and the "Dirichlet" fractional Laplacian by the quadratic form

$$Q^D_{s,\Omega}[u] = ((-\Delta_\Omega)^s_D u, u) := \int\limits_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi$$

with domains, respectively,

$$\mathrm{Dom}(Q^N_{s,\Omega}) = \{u \in \mathcal{D}'(\Omega) \, : \, Q^N_s[u] < \infty\};$$

$$\mathrm{Dom}(Q^D_{s,\Omega}) = \{u \in \mathcal{S}'(\mathbb{R}^n) \, : \, \mathrm{supp}\, u \subset \overline{\Omega}, \; Q^D_s[u] < \infty\}.$$

For $s = 1$, these two operators evidently coincide. We emphasize that, in contrast to $(-\Delta_\Omega)^s_N$, the operator $(-\Delta_\Omega)^s_D$ is not the sth power of the Dirichlet Laplacian for $s \neq 1$.

In the case $0 < s < 1$ both the operators $(-\Delta_\Omega)^s_N$ and $(-\Delta_\Omega)^s_D$ were considered in many articles on semilinear equations.

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Recall that the Sobolev space $H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the space of distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ with finite norm

$$\|u\|_s^2 = \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\mathcal{F}u(\xi)|^2 \, d\xi,$$

Also we introduce the space

$$\widetilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^n) \, : \, \operatorname{supp} u \subset \overline{\Omega}\}.$$

Note that $\widetilde{H}^s(\Omega)$ coincides with $H_0^s(\Omega)$ only for $s - \frac{1}{2} \notin \mathbb{Z}$.

Caffarelli and Silvestre (2007) connected the fractional Laplacian of order $\sigma \in (0,1)$ in $\mathbb{R}^n$ with generalized Dirichlet-to-Neumann map. In particular, for any $u \in \widetilde{H}^\sigma(\Omega)$ the function $w_\sigma^D(x,y)$ minimizing the weighted Dirichlet integral

$$\mathcal{E}_\sigma^D(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} |\nabla w(x,y)|^2 \, dxdy$$

over the set

$$\mathcal{W}_\sigma^D(u) = \Big\{ w(x,y) \, : \, \mathcal{E}_\sigma^D(w) < \infty \ , \quad w\Big|_{y=0} = u \Big\},$$

satisfies

$$Q_{\sigma,\Omega}^D[u] = C_\sigma \cdot \mathcal{E}_\sigma^D(w_\sigma^D). \tag{1}$$

Moreover, $w_\sigma^D(x,y)$ is the solution of the BVP

$$-\mathrm{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad w\Big|_{y=0} = u,$$

and for sufficiently smooth u

$$(-\Delta_\Omega)_D^\sigma u(x) = -C_\sigma \cdot \lim_{y \to 0^+} y^{1-2\sigma} \partial_y w_\sigma^D(x,y), \qquad x \in \Omega. \tag{2}$$

Stinga and Torrea (2010) developed this approach in quite general situation. In particular, it was shown that for any $u \in \widetilde{H}^\sigma(\Omega)$ the function $w_\sigma^N(x,y)$ minimizing the energy integral

$$\mathcal{E}_\sigma^N(w) = \int\limits_0^\infty \int\limits_\Omega y^{1-2\sigma} |\nabla w(x,y)|^2 \, dxdy$$

over the set

$$\mathcal{W}_{\sigma,\Omega}^N(u) = \{w(x,y) \in \mathcal{W}_\sigma^D(u) \,:\, w\Big|_{x \in \partial\Omega} = 0\},$$

satisfies

$$Q_{\sigma,\Omega}^N[u] = C_\sigma \cdot \mathcal{E}_\sigma^N(w_\sigma^N). \tag{3}$$

Moreover, $w_\sigma^N(x,y)$ is the solution of the BVP

$$-\mathrm{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+; \quad w\Big|_{y=0} = u; \quad w\Big|_{x \in \partial\Omega} = 0,$$

and for sufficiently smooth u it turns out that

$$(-\Delta_\Omega)_N^\sigma u(x) = -C_\sigma \cdot \lim_{y \to 0+} y^{1-2\sigma} \partial_y w_\sigma^N(x,y). \tag{4}$$

In a similar way, we connect negative fractional Laplacians of order $-\sigma \in (-1,0)$ with generalized Neumann-to-Dirichlet map. Namely, let $u \in \mathrm{Dom}(Q^D_{-\sigma,\Omega})$. Consider the problem of minimizing the functional

$$\widetilde{\mathcal{E}}^D_{-\sigma}(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} |\nabla w(x,y)|^2 \, dxdy \; - \; 2 \left\langle u, w\Big|_{y=0} \right\rangle$$

over the set $\mathcal{W}^D_{-\sigma}$, that is closure of smooth functions on $\mathbb{R}^n \times \bar{\mathbb{R}}_+$ with bounded support, with respect to $\mathcal{E}^D_\sigma(\cdot)$. We recall that u can be considered as a compactly supported functional on $H^\sigma(\mathbb{R}^n)$, and thus the duality $\left\langle u, w\Big|_{y=0} \right\rangle$ is well defined.

Denote the minimizer of $\widetilde{\mathcal{E}}^D_{-\sigma}$ by $w^D_{-\sigma}(x,y)$. Then formulae (1) and (2) imply relations

$$Q^D_{-\sigma,\Omega}[u] = -C_\sigma^{-1} \cdot \widetilde{\mathcal{E}}^D_{-\sigma}(w^D_{-\sigma}); \quad (-\Delta_\Omega)^{-\sigma}_D u(x) = C_\sigma^{-1} \, w^D_{-\sigma}(x,0), \quad x \in \Omega, \tag{5}$$

that give the “dual” variational characterization of $(-\Delta_\Omega)^{-\sigma}_D$.

Remark. Note that for sufficiently smooth u the function $w^D_{-\sigma}$ solves the Neumann problem

$$-\mathrm{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad \lim_{y\to 0+} y^{1-2\sigma}\partial_y w = -u.$$

Analogously, formulae (3) and (4) imply the "dual" variational characterization of $(-\Delta_\Omega)^{-\sigma}_N$. Namely, the function $w^N_{-\sigma}(x,y)$ minimizing the functional

$$\widetilde{\mathcal{E}}^N_{-\sigma}(w) = \int_0^\infty\int_\Omega y^{1-2\sigma}|\nabla w(x,y)|^2\,dxdy \, - \, 2\left\langle u, w\Big|_{y=0}\right\rangle$$

over the set

$$\mathcal{W}^N_{-\sigma,\Omega}(u) = \{w(x,y) \in \mathcal{W}^D_{-\sigma} \, : \, w\Big|_{x\notin\Omega} = 0\},$$

satisfies

$$Q^N_{-\sigma,\Omega}[u] = -C_\sigma^{-1}\cdot \widetilde{\mathcal{E}}^N_{-\sigma}(w^N_{-\sigma}); \quad (-\Delta_\Omega)^{-\sigma}_N u(x) = C_\sigma^{-1}\, w^N_{-\sigma}(x,0). \quad (6)$$

Theorem 1. Let $s > -1$, $s \notin \mathbb{N}_0$. Then for $u \in \mathrm{Dom}(Q^D_{s,\Omega})$, $u \not\equiv 0$, the following relations hold:

$$Q^N_{s,\Omega}[u] > Q^D_{s,\Omega}[u], \text{ if } 2k < s < 2k+1, \quad k \in \mathbb{N}_0; \tag{7}$$

$$Q^N_{s,\Omega}[u] < Q^D_{s,\Omega}[u], \text{ if } 2k-1 < s < 2k, \quad k \in \mathbb{N}_0. \tag{8}$$

1. Let $s = \sigma \in (0,1)$. We construct extensions w^D_σ and w^N_σ as described above.

We evidently have $\mathcal{W}^N_{\sigma,\Omega} \subset \mathcal{W}^D_\sigma$ and $\widetilde{\mathcal{E}}^N_\sigma = \widetilde{\mathcal{E}}^D_\sigma\Big|_{\mathcal{W}^N_{\sigma,\Omega}}$. Therefore, formulae (1) and (3) provide

$$Q^N_{s,\Omega}[u] = C_\sigma \cdot \inf_{w \in \mathcal{W}^N_{\sigma,\Omega}} \widetilde{\mathcal{E}}^N_\sigma(w) \geqslant C_\sigma \cdot \inf_{w \in \mathcal{W}^D_\sigma} \widetilde{\mathcal{E}}^D_\sigma(w) = Q^D_{s,\Omega}[u].$$

To complete the proof, we observe that for $u \not\equiv 0$ the function w^N_σ cannot be a solution of the homogeneous equation in the whole half-space, since such a solution is analytic in the half-space. Thus, it cannot provide $\inf\limits_{w \in \mathcal{W}^D_\sigma} \widetilde{\mathcal{E}}^D_\sigma(w)$, and (7) follows.

2. Let $-1 < s < 0$. We define $\sigma = -s \in (0,1)$ and construct extensions $w^D_{-\sigma}$ and $w^N_{-\sigma}$. All arguments above hold, but the inequality is reversed by the "$-$" sign in (5) and (6).

3. Now let $s > 1$, $s \notin \mathbb{N}$. We put $k = \lfloor \frac{s-1}{2} \rfloor$ and define for $u \in \widetilde{H}^s(\Omega)$

$$v = (-\Delta)^k u \in \widetilde{H}^{s-2k}(\Omega), \qquad s - 2k \in (-1,0) \cup (0,1).$$

Note that $v \not\equiv 0$ if $u \not\equiv 0$. Then we have

$$Q^N_{s,\Omega}[u] = Q^N_{s-2k,\Omega}[v], \qquad Q^D_{s,\Omega}[u] = Q^D_{s-2k,\Omega}[u],$$

and the conclusion follows from cases 1 and 2.

Remark. Frank and Geisinger (preprint, 2013) proved a general result which gives Theorem 1 for $s \in (0,1)$ with $\geqslant$ sign.

Next, we take into account the role of dilations in $\mathbb{R}^n$. We denote by $F(\Omega)$ the class of smooth and bounded domains containing Ω. If $\Omega' \in F(\Omega)$, then any $u \in \mathrm{Dom}(Q^D_{s,\Omega})$ can be regarded as a function in $\mathrm{Dom}(Q^D_{s,\Omega'})$, and the corresponding form $Q^D_{s,\Omega'}[u]$ does not change. In contrast, the form $Q^N_{s,\Omega'}[u]$ does depend on $\Omega' \supset \Omega$. However, roughly speaking, the difference between these quadratic forms disappears as $\Omega' \to \mathbb{R}^n$.

Theorem 2. Let $s > -1$. Then for $u \in \mathrm{Dom}(Q^D_{s,\Omega})$ the following facts hold:

$$Q^D_{s,\Omega}[u] = \inf_{\Omega' \in F(\Omega)} Q^N_{s,\Omega'}[u], \quad \text{if} \quad 2k < s < 2k+1, \quad k \in \mathbb{N}_0;$$

$$Q^D_{s,\Omega}[u] = \sup_{\Omega' \in F(\Omega)} Q^N_{s,\Omega'}[u], \quad \text{if} \quad 2k-1 < s < 2k, \quad k \in \mathbb{N}_0.$$

Remark. Assume that $0 \in \Omega$ and put $\alpha\Omega = \{\alpha x : x \in \Omega\}$. Then the proof shows indeed that

$$Q^D_{s,\Omega}[u] = \lim_{\alpha\to\infty} Q^N_{s,\alpha\Omega}[u] \quad \text{for any} \quad u \in \mathrm{Dom}(Q^D_{s,\Omega}).$$

Now put $u_\alpha(x) = \alpha^{\frac{n-2s}{2}} u(\alpha x)$. Then the scaling shows that

$$Q^D_{s,\Omega}[u_\alpha] \equiv Q^D_{s,\Omega}[u] = \lim_{\alpha\to\infty} Q^N_{s,\Omega}[u_\alpha] \quad \text{for any} \quad u \in \widetilde{H}^s(\Omega).$$

Moreover, this result was recently sharpened (RM & AN, 2015). Namely,

$$\left| Q^D_{s,\Omega}[u_\alpha] - Q^N_{s,\Omega}[u_\alpha] \right| = O(\alpha^{-(n+2s)}), \quad \text{as} \quad \alpha \to \infty.$$

Using this estimate we established the Brezis–Nitenberg type result for semilinear equations with Navier Laplacian and critical growth of the right-hand side.

We also obtain a pointwise comparison result.

Theorem 3. Let $0 < |s| < 1$, and let $u \in \mathrm{Dom}(Q^D_{s,\Omega})$, $u \geqslant 0$, $u \not\equiv 0$. Then the following relations hold:

$$(-\Delta_\Omega)^s_N u > (-\Delta_\Omega)^s_D u, \ \text{ if } \ 0 < s < 1;$$
$$(-\Delta_\Omega)^s_N u < (-\Delta_\Omega)^s_D u, \ \text{ if } \ -1 < s < 0.$$

Here all inequalities are understood in the sense of distributions.

Remark. Fall (preprint, 2012) proved this for $s = \frac{1}{2}$ and for smooth u.

We prove Theorem 3 for $s = \sigma \in (0, 1)$. First, let $u \in \mathcal{C}_0^\infty(\Omega)$. We construct extensions w_σ^D and w_σ^N described above. Since w_σ^D vanishes at infinity, $w_\sigma^D(x, t) > 0$ for $t > 0$ by the maximum principle. Then the strong maximum principle gives

$$W := w_\sigma^D - w_\sigma^N > 0 \qquad \text{in} \quad \Omega \times \mathbb{R}_+.$$

After changing of the variable $t = y^{2\sigma}$ the function $W(x, t)$ solves

$$\Delta_x W + 4s^2 t^{\frac{2s-1}{s}} W_{tt} = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+; \qquad W\Big|_{t=0} = 0. \tag{9}$$

The differential operator in (9) satisfies the assumptions of the boundary point lemma (Maz'ya et al., 2011) at any point $(x_0, 0) \in \Omega \times \{0\}$. Thus, we have

$$\liminf_{y \to 0^+} y^{1-2\sigma} \partial_y W(x, y) = 2\sigma \cdot \liminf_{t \to 0^+} \partial_t W(x, t) > 0.$$

For $u \in \widetilde{H}^s(\Omega)$ the statement holds by approximation argument.

The case $s < 0$ is managed in a similar way.