Cyclic polynomials in two variables

Daniel Seco (with C. Bénéteau, G. Knese, L. Kosiński, C. Liaw and A. Sola)

University of Warwick - Universitat de Barcelona

PPF, Linköping, 15/06/2015
$\mathbb{D} = \{ z : |z| < 1 \}$. The *Dirichlet-type space*, $D_\alpha$ may be defined as:

$$\{ f \in Hol(\mathbb{D}) : f(z) = \sum_{k \in \mathbb{N}} a_k z^k, \| f \|_\alpha^2 = \sum_{k \in \mathbb{N}} |a_k|^2 (k + 1)^\alpha < \infty \}$$
The space over the disc $\mathbb{D} = \{z : |z| < 1\}$. The Dirichlet-type space, $D_\alpha$ may be defined as:

$$\{f \in \text{Hol}(\mathbb{D}) : f(z) = \sum_{k \in \mathbb{N}} a_k z^k, \|f\|_\alpha^2 = \sum_{k \in \mathbb{N}} |a_k|^2 (k + 1)^\alpha < \infty\}$$

The (forward) shift operator: $S : D_\alpha \rightarrow D_\alpha$, defined by $Sf(z) = zf(z)$ is bounded.
\[ \mathbb{D} = \{ z : |z| < 1 \}. \text{ The Dirichlet-type space, } D_\alpha \text{ may be defined as:} \]

\[ \{ f \in \text{Hol}(\mathbb{D}) : f(z) = \sum_{k \in \mathbb{N}} a_k z^k, \|f\|_\alpha^2 = \sum_{k \in \mathbb{N}} |a_k|^2 (k + 1)^\alpha < \infty \} \]

- The (forward) shift operator: \( S : D_\alpha \to D_\alpha \), defined by \( Sf(z) = zf(z) \) is bounded.

- **Cycllicity class of** \( f \): \( [f]_\alpha (= [f]) = \text{span}\{ z^k f : k = 0, 1, 2, \ldots \} \). Polynomials dense subset of \( D_\alpha \Rightarrow [1] = D_\alpha \).
\( \mathbb{D} = \{ z : |z| < 1 \} \). The Dirichlet-type space, \( D_\alpha \) may be defined as:

\[
\{ f \in Hol(\mathbb{D}) : f(z) = \sum_{k \in \mathbb{N}} a_k z^k, \| f \|_\alpha^2 = \sum_{k \in \mathbb{N}} |a_k|^2 (k + 1)\alpha < \infty \}
\]

The (forward) shift operator: \( S : D_\alpha \to D_\alpha \), defined by \( Sf(z) = zf(z) \) is bounded.

Cyclicity class of \( f \): \([f]_\alpha = \text{span}\{ z^k f : k = 0, 1, 2, \ldots \}\).

Polynomials dense subset of \( D_\alpha \Rightarrow [1] = D_\alpha \).

A function \( f \) is cyclic (in \( D_\alpha \)) if \([f] = D_\alpha \) (\( \Leftrightarrow \exists \{ p_n \}_{n \in \mathbb{N}}, \) polynomials: \( \| p_n f - 1 \|_\alpha \xrightarrow{n \to \infty} 0 \))
Spaces over the disc

\[ \mathbb{D} = \{ z : |z| < 1 \}. \] The Dirichlet-type space, \( D_\alpha \) may be defined as:

\[ \{ f \in \text{Hol}(\mathbb{D}) : f(z) = \sum_{k \in \mathbb{N}} a_k z^k, \| f \|_\alpha^2 = \sum_{k \in \mathbb{N}} |a_k|^2 (k + 1)^\alpha < \infty \} \]

The (forward) shift operator: \( S : D_\alpha \to D_\alpha \), defined by \( Sf(z) = zf(z) \) is bounded.

**Cyclicity class of** \( f \): \([f]_\alpha (= [f]) = \text{span}\{ z^k f : k = 0, 1, 2, \ldots \} \). Polynomials dense subset of \( D_\alpha \Rightarrow [1] = D_\alpha \).

A function \( f \) is cyclic (in \( D_\alpha \)) if \([f] = D_\alpha \) (\( \iff \exists \{p_n\}_{n \in \mathbb{N}} \), polynomials: \( \|p_n f - 1\|_\alpha \xrightarrow{n \to \infty} 0 \)) (\( \iff \exists \{p_n\}_{n \in \mathbb{N}} \) polynomials: \( \|p_n f - 1\|_\alpha \leq C \) and \( p_n f \to 1 \) pw in \( \mathbb{D} \) (or unif. on comp.)).
Since polynomials are dense, 1 is cyclic for any $\alpha$. 
Since polynomials are dense, 1 is cyclic for any \( \alpha \).

If \( f \in M_{D_{\alpha}} \) and \( 1/f \in D_{\alpha} \) then \( f \) is cyclic in \( D_{\alpha} \) (for example if \( f \) and \( 1/f \) are holomorphic on a bigger disc).
Since polynomials are dense, 1 is cyclic for any $\alpha$.

If $f \in M_{D_\alpha}$ and $1/f \in D_\alpha$ then $f$ is cyclic in $D_\alpha$ (for example if $f$ and $1/f$ are holomorphic on a bigger disc).

If $\exists z_0 \in \mathbb{D}: f(z_0) = 0$, then $p_n$ can’t converge to $1/f$ at $z_0 \Rightarrow f$ is NOT cyclic for any $\omega$. 
Since polynomials are dense, 1 is cyclic for any $\alpha$.

If $f \in M_{D_\alpha}$ and $1/f \in D_\alpha$ then $f$ is cyclic in $D_\alpha$ (for example if $f$ and $1/f$ are holomorphic on a bigger disc).

If $\exists z_0 \in \mathbb{D} : f(z_0) = 0$, then $p_n$ can’t converge to $1/f$ at $z_0 \Rightarrow f$ is NOT cyclic for any $\omega$.

Beurling (’39): For Hardy space ($\alpha = 0$), cyclic iff outer.
Since polynomials are dense, 1 is cyclic for any $\alpha$.

If $f \in M_{D\alpha}$ and $1/f \in D\alpha$ then $f$ is cyclic in $D\alpha$ (for example if $f$ and $1/f$ are holomorphic on a bigger disc).

If $\exists z_0 \in \mathbb{D}: f(z_0) = 0$, then $p_n$ can’t converge to $1/f$ at $z_0 \Rightarrow f$ is NOT cyclic for any $\omega$.

Beurling (’39): For Hardy space ($\alpha = 0$), cyclic iff outer.

Brown-Shields (’84): $\alpha > 1$, cyclic $\iff 1/f \in H^\infty$. 
Since polynomials are dense, 1 is cyclic for any $\alpha$.

If $f \in M_{D\alpha}$ and $1/f \in D_{\alpha}$ then $f$ is cyclic in $D_{\alpha}$ (for example if $f$ and $1/f$ are holomorphic on a bigger disc).

If $\exists z_0 \in \mathbb{D}: f(z_0) = 0$, then $p_n$ can’t converge to $1/f$ at $z_0 \Rightarrow f$ is NOT cyclic for any $\omega$.

Beurling (’39): For Hardy space ($\alpha = 0$), cyclic iff outer.

Brown-Shields (’84): $\alpha > 1$, cyclic $\iff 1/f \in \mathbb{H}^\infty$.

In Dirichlet ($\alpha = 1$), neither nor: They also showed that critical polynomials are cyclic iff $\alpha \leq 1$. 
Since polynomials are dense, $1$ is cyclic for any $\alpha$.

If $f \in M_{D\alpha}$ and $1/f \in D\alpha$ then $f$ is cyclic in $D\alpha$ (for example if $f$ and $1/f$ are holomorphic on a bigger disc).

If $\exists z_0 \in \mathbb{D} : f(z_0) = 0$, then $p_n$ can't converge to $1/f$ at $z_0 \Rightarrow f$ is NOT cyclic for any $\omega$.

Beurling (’39): For Hardy space ($\alpha = 0$), cyclic iff outer.

Brown-Shields (’84): $\alpha > 1$, cyclic $\iff 1/f \in \mathbb{H}\infty$.

In Dirichlet ($\alpha = 1$), neither nor: They also showed that critical polynomials are cyclic iff $\alpha \leq 1$.

And they found a link with potential theory: cyclic implies $Z(f)$ small. Conjecture outer+small $Z(f)$ implies cyclic?
Since polynomials are dense, 1 is cyclic for any $\alpha$.

If $f \in M_{D_\alpha}$ and $1/f \in D_\alpha$ then $f$ is cyclic in $D_\alpha$ (for example if $f$ and $1/f$ are holomorphic on a bigger disc).

If $\exists z_0 \in \mathbb{D}: f(z_0) = 0$, then $p_n$ can’t converge to $1/f$ at $z_0 \Rightarrow f$ is NOT cyclic for any $\omega$.

Beurling (’39): For Hardy space ($\alpha = 0$), cyclic iff outer.

Brown-Shields (’84): $\alpha > 1$, cyclic $\iff 1/f \in \mathbb{H}_\infty$.

In Dirichlet ($\alpha = 1$), neither nor: They also showed that critical polynomials are cyclic iff $\alpha \leq 1$.

And they found a link with potential theory: cyclic implies $Z(f)$ small. Conjecture outer+small $Z(f)$ implies cyclic?

$\alpha > \alpha'$, cyclic in $D_\alpha \Rightarrow$ cyclic in $D_{\alpha'}$. 
BCLSS ('13): How cyclic is a function? How fast can that convergence be?
BCLSS ('13): How cyclic is a function? How fast can that convergence be? That is, if we fix the degree of $p_n$ to be $\leq n$, how fast can $\|p_nf - 1\|_\alpha^2$ go to zero?
BCLSS ('13): How cyclic is a function? How fast can that convergence be? That is, if we fix the degree of $p_n$ to be $\leq n$, how fast can $\|p_n f - 1\|_2^2$ go to zero?

Optimizational viewpoint: Define $V_n = \{pf : p poly, \deg p \leq n\}$ and $\Pi_n$ orthogonal proj. of $D_\alpha$ onto $V_n$. 
BCLSS ('13): How cyclic is a function? How fast can that convergence be? That is, if we fix the degree of \( p_n \) to be \( \leq n \), how fast can \( \|p_n f - 1\|_\alpha^2 \) go to zero?

Optimizational viewpoint: Define \( V_n = \{pf : p \text{ poly}, \deg p \leq n\} \) and \( \Pi_n \) orthogonal proj. of \( D_\alpha \) onto \( V_n \).

Then \( \exists! \Pi_n(1) \), best approximation to 1 in \( V_n \). We call the best approximant to \( 1/f \) of degree \( n \) to the polynomial \( p^*_n : p^*_n f = \Pi_n(1) \).
BCLSS (’13): How cyclic is a function? How fast can that convergence be? That is, if we fix the degree of $p_n$ to be $\leq n$, how fast can $\|p_n f - 1\|_\alpha^2 \to 0$?

Optimizational viewpoint: Define $V_n = \{p f : p \in \text{poly}, \deg p \leq n\}$ and $\Pi_n$ orthogonal proj. of $D_\alpha$ onto $V_n$.

Then $\exists! \Pi_n(1)$, best approximation to 1 in $V_n$. We call the best approximant to $1/f$ of degree $n$ to the polynomial $p_n^*: p_n^* f = \Pi_n(1)$.

With this, cyclic $\iff \|p_n^* f - 1\|_\alpha^2 \to 0 \iff p_n^* \to 1/f$ pw in $\mathbb{D} \iff p_n^* \to 1/f$ unif. on comp.
We solved the optimization problems:

\[ p^*_n(z) = \sum_{k=0}^{n-1} c_k z^k \]

only solution to \( Mc = b \) where 
\[ c = (c_k), \quad M_{i,j} = \begin{cases} 1 & i = j, \\ \alpha^{z_i} & z_i < z_j, \\ 1 & z_j < z_i, \end{cases} \]
\[ b_j = \langle 1, z_{j+1} \rangle \]

Simplest critical case, \( f(z) = 1 - z \), closed general formula for \( p^*_n \) for all \( n \).
We solved the optimization problems:

**Theorem**

\[ p_n^*(z) = \sum_{k=0}^{n} c_k z^k \text{ only solution to } Mc = b \text{ where} \]

\[ c = (c_k)_{k=0}^{n}, \quad M_{i,j} = \langle z^i f, z^j f \rangle_{\alpha}, \quad b_j = \langle 1, z^j f \rangle_{\alpha}. \]
We solved the optimization problems:

**Theorem**

\[ p_n^*(z) = \sum_{k=0}^{n} c_k z^k \] is the only solution to \( Mc = b \) where

\[ c = (c_k)_{k=0}^{n}, \quad M_{i,j} = \langle z^i f, z^j f \rangle_{\alpha}, \quad b_j = \langle 1, z^j f \rangle_{\alpha}. \]

Simplest critical case, \( f(z) = 1 - z \), closed general formula for \( p_n^* \) for all \( n \).
Quantitative bounds to the speed of convergence of $\|p_n f - 1\|_2^2$ to 0 for $f \in Hol(D)$:
Cyclicity
Asymptotic quantitative behavior

Quantitative bounds to the speed of convergence of $\|p_n^*f - 1\|_2^2$ to 0 for $f \in Hol(\overline{D})$:

**Theorem**

$f \in Hol(\overline{D}), Z(f) \cap D = \emptyset \neq Z(f) \cap T$. Then

$$\|p_n^*f - 1\|_2^2 \approx \frac{1}{\sum_{j=0}^{n+1} (j + 1)^{-\alpha}}$$
Quantitative bounds to the speed of convergence of $\|p_n^* f - 1\|_2^2$ to 0 for $f \in Hol(\overline{D})$:

**Theorem**

$f \in Hol(\overline{D}), Z(f) \cap \mathbb{D} = \emptyset \neq Z(f) \cap \mathbb{T}$. Then

$$\|p_n^* f - 1\|_2^2 \approx \frac{1}{\sum_{j=0}^{n+1} (j + 1)^{-\alpha}}$$

1 – $z$ is the key: FTA + cyclic polynomials iff cyclic factors + finding explicitly polys for $1 - z$. 
Useful techniques for studying cyclicity of a fixed function.
Further
Where to from here?

- Useful techniques for studying cyclicity of a fixed function.
- Between $Z(p^*_n)$ and $\lim p^*_n(0)$, we have all the information of the cyclicity of a function. Can the asymptotic properties of $Z(p^*_n)$ ($n \to \infty$) tell us something about capacities, potentials?
Useful techniques for studying cyclicity of a fixed function. Between $Z(p_n^*)$ and $\lim p_n^*(0)$, we have all the information of the cyclicity of a function. Can the asymptotic properties of $Z(p_n^*)$ ($n \to \infty$) tell us something about capacities, potentials? What about changing $\mathbb{D}$ by $\mathbb{D}^2$?
Our choice: the bidisc $\mathbb{D}^2 = \{ z_1, z_2 : |z_i| < 1 \}$. The Dirichlet-type space over the bidisc, $D_\alpha$ may be defined as:

$$\{ f(z_1, z_2) = \sum_{k,l \in \mathbb{N}} a_{k,l} z_1^k z_2^l, \| f \|_\alpha^2 = \sum |a_{k,l}|^2 (k+1)^\alpha (l+1)^\alpha < \infty \}$$
Our choice: the bidisc $\mathbb{D}^2 = \{z_1, z_2 : |z_i| < 1\}$. The Dirichlet-type space over the bidisc, $D_\alpha$ may be defined as:

$$\{ f(z_1, z_2) = \sum_{k,l \in \mathbb{N}} a_{k,l} z_1^k z_2^l, \|f\|_\alpha^2 = \sum |a_{k,l}|^2 (k + 1)^\alpha (l + 1)^\alpha < \infty \}$$

Polynomials are in two variables, degree is sum of partial degrees, or maximum... but no FTA: many irreducible polynomials!
Our choice: the bidisc $\mathbb{D}^2 = \{z_1, z_2 : |z_i| < 1\}$. The *Dirichlet-type space over the bidisc, $D_\alpha$* may be defined as:

$$\{f(z_1, z_2) = \sum_{k,l \in \mathbb{N}} a_{k,l} z_1^k z_2^l, \|f\|_\alpha^2 = \sum |a_{k,l}|^2 (k + 1)^\alpha (l + 1)^\alpha < \infty\}$$

Polynomials are in two variables, degree is sum of partial degrees, or maximum... but no FTA: many irreducible polynomials!

Rudin (’69): In Hardy, cyclic implies outer but outer does not imply cyclic. Neuwirth-Ginsberg (’70) and Gelca (’95): At least when $f$ is a polynomial, outer implies cyclic.
Two variables

Yeah, what about that?

- Our choice: the bidisc $\mathbb{D}^2 = \{z_1, z_2 : |z_i| < 1\}$. The *Dirichlet-type space over the bidisc*, $D_\alpha$, may be defined as:

  \[
  \{f(z_1, z_2) = \sum_{k,l \in \mathbb{N}} a_{k,l} z_1^k z_2^l, \; \|f\|_\alpha^2 = \sum |a_{k,l}|^2 (k + 1)^\alpha (l + 1)^\alpha < \infty\}
  \]

- Polynomials are in two variables, degree is sum of partial degrees, or maximum... but no FTA: many irreducible polynomials!
- Rudin ('69): In Hardy, cyclic implies outer but outer does not imply cyclic. Neuwirth-Ginsberg ('70) and Gelca ('95): At least when $f$ is a polynomial, outer implies cyclic.
- When $\alpha > 1$, Hedenmalm ('88) same as in 1 variable. In Bergman ($\alpha = -1$), Massaneda-Thomas ('13): there is no characterization in terms of growth.
In a first work in 2014, we (BCLSS) showed many of Brown-Shields and our own results work in 2 variables:

- Cyclic in $D_\alpha$ implies small $Z(f)$.
In a first work in 2014, we (BCLSS) showed many of Brown-Shields and our own results work in 2 variables:

- Cyclic in $D_\alpha$ implies small $Z(f)$.
- Optimal polynomials approach works.
Our work on polynomials
BCLSS2

In a first work in 2014, we (BCLSS) showed many of Brown-Shields and our own results work in 2 variables:

- Cyclic in $D_\alpha$ implies small $Z(f)$.
- Optimal polynomials approach works.
- Some natural although trivial properties are extended from 1 variable (products, restrictions, extensions...).
In a first work in 2014, we (BCLSS) showed many of Brown-Shields and our own results work in 2 variables:

- Cyclic in $D_\alpha$ implies small $Z(f)$.
- Optimal polynomials approach works.
- Some natural although trivial properties are extended from 1 variable (products, restrictions, extensions...).

But we also showed polynomials are more complicated! There are at least 4 cases: $(1 - z_1 - z_2)$, $(2 - z_1 - z_2)$, $(3 - z_1 - z_2)$, $1 - z_1 z_2$ are all different.
In our work BKKLSS('14) we have characterized now cyclicity for polynomials in $D_\alpha(D^2)$:

**Theorem**

*Let $f$ be a polynomial in two variables. Then there are exactly 4 cases:*

1. $Z(f) \cap D^2 = \emptyset = \Rightarrow f$ not cyclic in $D_\alpha(D^2)$.
2. $Z(f) \cap D^2 = \emptyset = \Rightarrow f$ is cyclic in $D_\alpha(D^2)$.
3. $Z(f) \cap D^2 = Z(f) \cap T^2$ small but $\neq \emptyset = \Rightarrow f$ cyclic iff $\alpha \leq 1$.
4. Otherwise $Z(f) \cap D^2$ contains a transversal curve over $T^2$ and $f$ is cyclic iff $\alpha \leq 1/2$.

Small for polynomials, well understood: contained on finitely many vert. or hor. slices.

All builds on work of Gelca, Hedenmalm, Lojasiewicz and constructing an integration current and a determinantal representation for stable polynomials due to Knese ('10).
In our work BKKLSS('14) we have characterized now cyclicity for polynomials in $D_\alpha(\mathbb{D}^2)$:

**Theorem**

Let $f$ be a polynomial in two variables. Then there are exactly 4 cases:

1. $Z(f) \cap \mathbb{D}^2 \neq \emptyset \implies f$ not cyclic in $D_\alpha$.
2. $Z(f) \cap \mathbb{D}^2 = \emptyset \implies f$ cyclic in $D_\alpha$.
3. $Z(f) \cap \mathbb{D}^2 = Z(f) \cap T^2$ small but $\neq \emptyset \implies$ cyclic iff $\alpha \leq 1$.
4. Otherwise $Z(f) \cap \mathbb{D}^2$ contains a transversal curve over $T^2$ and $f$ is cyclic iff $\alpha \leq 1/2$.

Small for polynomials, well understood: contained on finitely many vert. or hor. slices.
Polynomials in two variables
Classification of cyclicity 1

In our work BKKLSS('14) we have characterized now cyclicity for polynomials in $D_\alpha(D^2)$:

**Theorem**

Let $f$ be a polynomial in two variables. Then there are exactly 4 cases:

1. $Z(f) \cap D^2 \neq \emptyset \implies f \text{ not cyclic in } D_\alpha$.
2. $Z(f) \cap \overline{D^2} = \emptyset \implies f \text{ is cyclic in } D_\alpha$.

Small for polynomials, well understood: contained on finitely many vert. or hor. slices. All builds on work of Gelca, Hedenmalm, Lojasiewicz and constructing an integration current and a determinantal representation for stable polynomials due to Knese ('10).
Polynomials in two variables

Classification of cyclicity 1

In our work BKKLSS('14) we have characterized now cyclicity for polynomials in $D_\alpha(D^2)$:

**Theorem**

Let $f$ be a polynomial in two variables. Then there are exactly 4 cases:

1. $\mathcal{Z}(f) \cap D^2 \neq \emptyset \implies f$ not cyclic in $D_\alpha$.
2. $\mathcal{Z}(f) \cap D^2 = \emptyset \implies f$ is cyclic in $D_\alpha$.
3. $\mathcal{Z}(f) \cap D^2 = \mathcal{Z}(f) \cap \mathbb{T}^2$ small but $\neq \emptyset \implies f$ cyclic iff $\alpha \leq 1$.
4. Otherwise $\mathcal{Z}(f) \cap D^2$ contains a transversal curve over $\mathbb{T}^2$ and $f$ is cyclic iff $\alpha \leq 1/2$.

Small for polynomials, well understood: contained on finitely many vert. or hor. slices.

All builds on work of Gelca, Hedenmalm, Lojasiewicz and constructing an integration current and a determinantal representation for stable polynomials due to Knese ('10).
In our work BKKLSS('14) we have characterized now cyclicity for polynomials in $D_\alpha(D^2)$:

**Theorem**

Let $f$ be a polynomial in two variables. Then there are exactly 4 cases:

1. $Z(f) \cap D^2 \neq \emptyset \implies f$ not cyclic in $D_\alpha$.
2. $Z(f) \cap \overline{D^2} = \emptyset \implies f$ is cyclic in $D_\alpha$.
3. $Z(f) \cap D^2 = Z(f) \cap T^2$ small but $\neq \emptyset \implies f$ cyclic iff $\alpha \leq 1$.
4. Otherwise $Z(f) \cap \overline{D^2}$ contains a transversal curve over $T^2$ and $f$ is cyclic iff $\alpha \leq 1/2$. 
In our work BKKLSS('14) we have characterized now cyclicity for polynomials in $D_\alpha(D^2)$:

**Theorem**

Let $f$ be a polynomial in two variables. Then there are exactly 4 cases:

1. $Z(f) \cap D^2 \neq \emptyset \implies f$ not cyclic in $D_\alpha$.
2. $Z(f) \cap D^2 = \emptyset \implies f$ is cyclic in $D_\alpha$.
3. $Z(f) \cap \overline{D^2} = Z(f) \cap \mathbb{T}^2$ small but $\neq \emptyset \implies f$ cyclic iff $\alpha \leq 1$.
4. Otherwise $Z(f) \cap \overline{D^2}$ contains a transversal curve over $\mathbb{T}^2$ and $f$ is cyclic iff $\alpha \leq 1/2$.

Small for polynomials, well understood: contained on finitely many vert. or hor. slices.
In our work BKKLSS(’14) we have characterized now cyclicity for polynomials in $D_\alpha(D^2)$:

**Theorem**

Let $f$ be a polynomial in two variables. Then there are exactly 4 cases:

1. $Z(f) \cap D^2 \neq \emptyset \implies f$ not cyclic in $D_\alpha$.
2. $Z(f) \cap D^2 = \emptyset \implies f$ is cyclic in $D_\alpha$.
3. $Z(f) \cap \overline{D^2} = Z(f) \cap \mathbb{T}^2$ small but $\neq \emptyset \implies f$ cyclic iff $\alpha \leq 1$.
4. Otherwise $Z(f) \cap \overline{D^2}$ contains a transversal curve over $\mathbb{T}^2$ and $f$ is cyclic iff $\alpha \leq 1/2$.

Small for polynomials, well understood: contained on finitely many vert. or hor. slices. All builds on work of Gelca, Hedenmalm, Lojasiewicz and constructing an integration current and a determinantal representation for stable polynomials due to Knese (’10).
Thank you!