Cyclic polynomials in two variables

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PPF, Linköping, 15/06/2015

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Introduction

Examples and motivation

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- $\alpha > \alpha'$, cyclic in $D_\alpha \Rightarrow$ cyclic in $D_{\alpha'}$.

Cyclicity

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- With this, cyclic $\Leftrightarrow ||p_n^* f - 1||_\alpha^2 \to 0 \Leftrightarrow p_n^* \to 1/f$ pw in $\mathbb{D} \Leftrightarrow p_n^* \to 1/f$ unif. on comp.

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Theorem

$p_n^*(z) = \sum_{k=0}^{n} c_k z^k$ *only solution to* $Mc = b$ *where*

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- Simplest critical case, $f(z) = 1 - z$, closed general formula for p_n^* for all n.

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$f \in Hol(\overline{\mathbb{D}})$, $Z(f) \cap \mathbb{D} = \emptyset \neq Z(f) \cap \mathbb{T}$. Then

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$1 - z$ is the key: FTA + cyclic polynomials iff cyclic factors + finding explicitly polys for $1 - z$.

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- Useful techniques for studying cyclicity of a fixed function.
- Between $Z(p_n^*)$ and $\lim p_n^*(0)$, we have all the information of the cyclicity of a function. Can the asymptotic properties of $Z(p_n^*)$ ($n \to \infty$) tell us something about capacities, potentials?
- What about changing $\mathbb{D}$ by $\mathbb{D}^2$?

Two variables 1

Yeah, what about that?

- Our choice: the bidisc $\mathbb{D}^2 = \{z_1, z_2 : |z_i| < 1\}$. The *Dirichlet-type space over the bidisc*, D_α may be defined as:

$$\{f(z_1, z_2) = \sum_{k,l \in \mathbb{N}} a_{k,l} z_1^k z_2^l, ||f||_\alpha^2 = \sum |a_{k,l}|^2 (k+1)^\alpha (l+1)^\alpha < \infty\}$$

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- When $\alpha > 1$, Hedenmalm ('88) same as in 1 variable. In Bergman ($\alpha = -1$), Massaneda-Thomas ('13): there is no characterization in terms of growth.

Our work on polynomials

BCLSS2

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But we also showed polynomials are more complicated! There are at least 4 cases: $(1 - z_1 - z_2)$, $(2 - z_1 - z_2)$, $(3 - z_1 - z_2)$, $1 - z_1 z_2$ are all different.

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The End

Thank you!