Compactness results for limiting interpolation methods

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PDE's, Potential Theory, Function Spaces
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Theorem (Riesz-Thorin). Let (R, μ) and (S, ν) be σ-finite measure spaces. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and a linear operator T such that

$$T : L_{p_0}(R, \mu) \longrightarrow L_{q_0}(S, \nu) \quad \text{with norm } M_0 \text{ and}$$
$$T : L_{p_1}(R, \mu) \longrightarrow L_{q_1}(S, \nu) \quad \text{with norm } M_1.$$

Then $T : L_p(R, \mu) \longrightarrow L_q(S, \nu)$ with norm $M \le M_0^{1-\theta} M_1^{\theta}$ whenever $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

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$A \hookrightarrow \mathcal{A}$ is intermediate if $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$. It is an interpolation space if, whenever $T \in \mathcal{L}(A_0 + A_1, A_0 + A_1)$ is such that $T \in \mathcal{L}(A_0, A_0)$ and $T \in \mathcal{L}(A_1, A_1)$, then $T \in \mathcal{L}(A, A)$.

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Peetre's K- and J-functionals: For every $t > 0$,

$$K(t, a; A_0, A_1) = \inf \left\{ \|a_0\|_{A_0} + t \|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j \right\}, \qquad a \in A_0 + A_1,$$

$$J(t, a) = \max\left(\|a\|_{A_0}, t \|a\|_{A_1} \right), \quad a \in A_0 \cap A_1.$$

(Classical) real interpolation spaces: Let $0 < \theta < 1$ and $1 \leq q \leq \infty$.

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{\theta,q} = \left(\int_0^\infty \left[t^{-\theta} K(t,a) \right]^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

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Given a function f, let $f^*(t) = \inf\{s > 0 : \mu(\{x \in \Omega : |f(x)| > s\}) \leq t\}$.

$\rightarrow$Lorentz spaces $L_{p,q}(\Omega)$. Ω σ-finite, $1 \leq p, q \leq \infty$.

$$\|f\|_{p,q} = \left(\int_0^\infty \left[t^{1/p} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q} \rightsquigarrow (L_\infty, L_1)_{\theta,q} = L_{1/\theta,q}$$

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→Lorentz-Zygmund spaces $L_{p,q}(\log L)_b(\Omega)$. Ω σ-finite, $1 \le p, q \le \infty$, $b \in \mathbb{R}$.

$$\|f\|_{p,q,b} = \left(\int_0^\infty \left[t^{1/p} (1 + |\log t|)^b f^*(t) \right]^q \frac{dt}{t} \right)^{1/q}.$$

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$$(L_\infty, L_1)_{1/p,\rho_b,q} = L_{p,q}(\log L)_b, \text{ where } \rho_b(t) = (1 + |\log t|)^b.$$

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What if $\theta = 0$ or $\theta = 1$?

Butzer, Berens, Springer-Verlag, 1967:
One can take $0 \le \theta \le 1$ whenever $q = \infty$ for the K-method and whenever $q = 1$ for the J-method.

$$(A_0, A_1)_{\theta,g,q} = \left\{ a \in A_0 + A_1 : \|a\|_{\theta,g,q} = \left(\int_0^\infty \left[t^{-\theta} g(t) K(t,a) \right]^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

In these cases, θ can take the values $0, 1$, but $g(t)$ is essential to get a meaningful definition.

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$$\|f\|_{L^{p)}} = \sup_{0<t<1} (1 + |\log t|)^{-1/p} \left(\int_t^1 [f^*(s)]^p ds \right)^{1/p},$$

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$$L_p(\log L)_{-1/p} \hookrightarrow L^{p)} \hookrightarrow L_p(\log L)_{-1/p-\delta}, \qquad \delta > 0.$$

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A. Fiorenza, G. E. Karadzhov, J. Anal. Appl. 23 (2004), 657–681:

$$\|f\|_{L^{p)}} \sim \sup_{0<t<\infty} (1 + |\log t|)^{-1/p} K(t, f; L_\infty, L_1).$$

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Let $\ell(t) = 1 + |\log t|$; for $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ write

$$\ell^{\mathbb{A}}(t) = \ell^{(\alpha_0,\alpha_\infty)}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{if } 0 < t \leq 1, \\ \ell^{\alpha_\infty}(t) & \text{if } 1 < t < \infty. \end{cases}$$

For $0 \leq \theta \leq 1$ and $1 \leq q \leq \infty$,

$$(A_0, A_1)_{\theta,q,\mathbb{A}} = \left\{ a \in A_0 + A_1 : \|a\|_{\theta,q,\mathbb{A}} = \left(\int_0^\infty \left[t^{-\theta} \ell^{\mathbb{A}}(t) K(t,a) \right]^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

Limiting methods in the ordered case

$A_0 \hookrightarrow A_1$.

M. E. Gomez, M. Milman, J. London Math. Soc. (1986) 305–316.

$$\bar{A}_{1,q;K} = \left\{ a \in A_1 : \|a\|_{1,q;K} = \left(\int_1^\infty \left[t^{-1} K(t,a) \right]^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

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B. Jawerth, M. Milman, Mem. Amer. Soc. 440 (1991): If Ω is finite,

$$\|f\|_{L(\log L)} \sim \int_1^\infty t^{-1} K(t,f;L_\infty,L_1) \frac{dt}{t}.$$

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F. Cobos, L. M. Fernández-Cabrera, T. Kühn, T. Ullrich, J. Funct. Anal. (2009) 2321–2366.

$$\bar{A}_{0,q;J} = \left\{ a \in A_1 : a = \int_1^\infty u(t) \frac{dt}{t} \text{ with } \left(\int_1^\infty J(t, u(t))^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

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An application

Let $F(f) = (\hat{f}(m))$. It is well-known that

$$F : L_1([0, 2\pi]) \longrightarrow \ell_\infty \qquad \text{and}$$

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Interpolating by the $(1,q;K)$-method, Gomez and Milman obtained that

$$F : L(\log L)([0,2\pi]) \longrightarrow \ell_1 \qquad \text{(Hardy-Littlewood), and}$$
$$F : L(\log L)_q([0,2\pi]) \longrightarrow \ell_1(\log \ell)_{1/q'},\ q > 0 \qquad \text{(Bennett).}$$

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Interpolating by the $(0,2;J)$-method,

$$F : L_2(\log L)_{-1/2}([0,2\pi]) \longrightarrow \ell_{2,\infty}(\log \ell)_{-1/2}.$$

Limiting interpolation methods
$(\theta = 0, \theta = 1)$

- Ordered setting $(A_0 \hookrightarrow A_1)$
  - $\theta = 0 \rightsquigarrow \bar{A}_{0,q;J}$
  - $\theta = 1 \rightsquigarrow \bar{A}_{1,q;K}$
- Function parameter $\rightsquigarrow \bar{A}_{\theta,g,q}$

Limiting K-spaces for general couples

Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. The space $\bar{A}_{q;K} = (A_0, A_1)_{q;K}$ consists of the vectors $a \in A_0 + A_1$ for which the following norm is finite

$$\|a\|_{\bar{A}_{q;K}} = \left(\int_0^1 K(t,a)^q \frac{dt}{t} \right)^{1/q} + \left(\int_1^\infty \left[t^{-1} K(t,a) \right]^q \frac{dt}{t} \right)^{1/q}.$$

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Classical definition $\|a\|_{\theta,q} = \left(\int_0^\infty \left[t^{-\theta} K(t,a) \right]^q \frac{dt}{t} \right)^{1/q}$.

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$\rightarrow \bar{A}_{q;K}$ extends $\bar{A}_{1,q;K}$ to general couples and $(A_0, A_1)_{q;K} = (A_1, A_0)_{q;K}$.

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→ $(A_0, A_1)_{\infty;K} = A_0 + A_1$.

Limiting J-spaces for general couples

Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. The space $\bar{A}_{q;J} = (A_0, A_1)_{q;J}$ is defined as the collection of vectors $a \in A_0 + A_1$ for which there exists a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty u(t) \frac{dt}{t} \qquad \text{(convergence in } A_0 + A_1\text{)} \tag{1}$$

and

$$\left(\int_0^1 \left[t^{-1} J(t, u(t)) \right]^q \frac{dt}{t} \right)^{1/q} + \left(\int_1^\infty J(t, u(t))^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{2}$$

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$$\left(\int_0^1 \left[t^{-1} J\left(t, u(t)\right)\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty J\left(t, u(t)\right)^q \frac{dt}{t}\right)^{1/q} < \infty. \tag{2}$$

The norm $\|a\|_{\bar{A}_{q;J}}$ is the infimum in (2) over all possible representations (1) of a.

Limiting J-spaces for general couples

Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. The space $\bar{A}_{q;J} = (A_0, A_1)_{q;J}$ is defined as the collection of vectors $a \in A_0 + A_1$ for which there exists a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty u(t) \frac{dt}{t} \qquad \text{(convergence in } A_0 + A_1\text{)} \tag{1}$$

and

$$\left(\int_0^1 \left[t^{-1} J(t, u(t)) \right]^q \frac{dt}{t} \right)^{1/q} + \left(\int_1^\infty J(t, u(t))^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{2}$$

The norm $\|a\|_{\bar{A}_{q;J}}$ is the infimum in (2) over all possible representations (1) of a.

Classical definition $\left(\int_0^\infty \left[t^{-\theta} J(t, u(t)) \right]^q \frac{dt}{t} \right)^{1/q} < \infty$.

→ $\bar{A}_{q;J}$ extends $\bar{A}_{0,q;J}$ to general couples and $(A_0, A_1)_{q;J} = (A_1, A_0)_{q;J}$.

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Theorem. For any $1 \le p, q, r \le \infty$ and $0 < \theta < 1$ we have that

$$\begin{array}{ccccccccc} A_0 \cap A_1 & \hookrightarrow & (A_0, A_1)_{p;J} & \hookrightarrow & (A_0, A_1)_{\theta,q} & \hookrightarrow & (A_0, A_1)_{r;K} & \hookrightarrow & A_0 + A_1 \\ \| & & & & & & & & \| \\ (A_0, A_1)_{1;J} & & & & & & & & (A_0, A_1)_{\infty;K} \end{array}$$

An example: (L_∞, L_1)

Lorentz-Zygmund spaces $L_{p,q}(\log L)_b(\Omega)$.

$$\|f\|_{p,q} = \left(\int_0^\infty \left[t^{1/p} (1 + |\log t|)^b f^*(t) \right]^q \frac{dt}{t} \right)^{1/q}.$$

$L_{(p,q)}(\log L)_b(\Omega)$ is defined replacing f^* by $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$.

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→If $1 \leq q \leq \infty$ and $0 < \theta < 1$ then $(L_\infty(\Omega), L_1(\Omega))_{\theta,q} = L_{1/\theta,q}(\Omega)$.

→In the limiting ordered case, if (Ω, μ) is a finite measure space then

$$(L_\infty(\Omega), L_1(\Omega))_{0,q;J} = L_{\infty,q}(\log L)_{-1}(\Omega) \quad \text{([CFKU])}.$$

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→In the limiting case for general couples, if (Ω, μ) is a σ-finite measure space then

$$(L_\infty(\Omega), L_1(\Omega))_{q;J} = L_{\infty,q}(\log L)_{-1}(\Omega) \cap L_{(1,q)}(\log L)_{-1}(\Omega).$$

Interpolation of compact operators

Theorem (Krasnosel'skiĭ). Let (R, μ) and (S, ν) be σ-finite measure spaces and let $1 \le p_0, p_1, q_0, q_1 \le \infty$, with $q_0 < \infty$, and consider a linear operator T such that

$$\begin{aligned} T : L_{p_0}(R, \mu) &\longrightarrow L_{q_0}(S, \nu) \quad \text{is compact and} \\ T : L_{p_1}(R, \mu) &\longrightarrow L_{q_1}(S, \nu) \quad \text{is bounded.} \end{aligned}$$

Then $T : L_p(R, \mu) \longrightarrow L_q(S, \nu)$ is compact if $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Interpolation of compact operators

Classical setting. 1992, M. Cwikel; F. Cobos, T. Kühn, T. Schonbek:

Interpolation of compact operators

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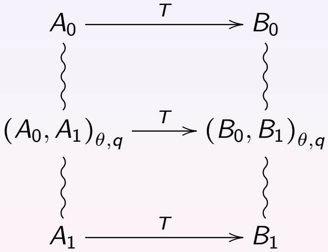

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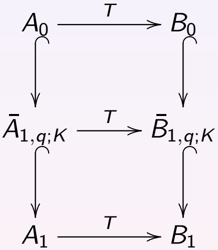

Limiting methods in the ordered setting ($A_0 \hookrightarrow A_1$, $B_0 \hookrightarrow B_1$) ([CFKU]):

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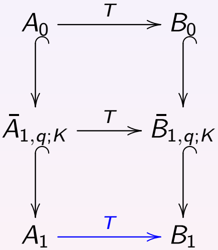

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- If $T : A_1 \to B_1$ is compact, then $T : \bar{A}_{1,q;K} \to \bar{B}_{1,q;K}$ is also compact, whereas the compactness of $T : A_0 \to B_0$ is not sufficient.

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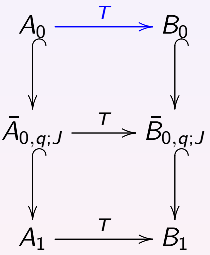

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- If $T : A_0 \to B_0$ is compact, then $T : \bar{A}_{0,q;J} \to \bar{B}_{0,q;J}$ is also compact, whereas the compactness of $T : A_1 \to B_1$ is not sufficient.

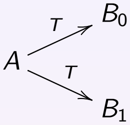
Limiting methods in the general setting. The compactness of one restriction is not sufficient.

Limiting methods in the general setting. The compactness of one restriction is not sufficient.

Theorem. Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples, let $T \in \mathcal{L}(\bar{A}, \bar{B})$ and let $1 \le q \le \infty$. If $T : A_j \to B_j$ is compact for $j = 0$ and 1, then so are

$$T : \bar{A}_{q;K} \to \bar{B}_{q;K} \text{ and}$$
$$T : \bar{A}_{q;J} \to \bar{B}_{q;J}.$$

- Let A be a Banach space, let $\bar{B} = (B_0, B_1)$ be a Banach couple and let $1 \leq q \leq \infty$. If T is a linear operator such that $T : A \to B_j$ is continuous for $j = 0, 1$



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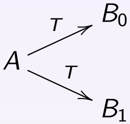

and any of the restrictions is compact, then $T : A \to \bar{B}_{q;K}$ is also compact.

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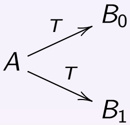

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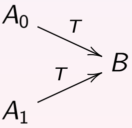

and any of the restrictions is compact, then $T : \bar{A}_{q;J} \to B$ is also compact.

L. M. Fernández-Cabrera, A. Martínez, Studia Math. (2014) 187–196 :

- Let A be a Banach space, let $\bar{B} = (B_0, B_1)$ be a Banach couple and let $1 \leq q \leq \infty$. If T is a linear operator such that $T : A \to B_0 + B_1$ is compact, then so is $T : A \to \bar{B}_{q;K}$.

- Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let B be a Banach space and let $1 \leq q \leq \infty$. If T is a linear operator such that $T : A_0 \cap A_1 \to B$ is compact, then so is $T : \bar{A}_{q;J} \to B$.

Logarithmic interpolation methods

Put $\ell(t) = 1 + |\log t|$; for $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ write

$$\ell^{\mathbb{A}}(t) = \ell^{(\alpha_0, \alpha_\infty)}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{if } 0 < t \leq 1, \\ \ell^{\alpha_\infty}(t) & \text{if } 1 < t < \infty. \end{cases}$$

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For $0 \le \theta \le 1$ and $1 \le q \le \infty$,

$$(A_0, A_1)_{\theta,q,\mathbb{A}} = \left\{ a \in A_0 + A_1 : \|a\|_{\theta,q,\mathbb{A}} = \left(\int_0^\infty \left[t^{-\theta} \ell^{\mathbb{A}}(t) K(t,a) \right]^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

J. Gustavsson, Math. Scand. 42 (1978), 289–305

R. Ya. Doktorskii, Soviet Math. Dokl. 44 (1992), 665–669

W. D. Evans, B. Opic, Canad. J. Math. 52 (2000), 920–960

W. D. Evans, B. Opic, L. Pick, J. Inequal. Appl. 7 (2002), 187–269.

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D. E. Edmunds, B. Opic, J. Funct. Anal. 266 (2014), 3265–3285.

If $A_0 \hookrightarrow A_1$ then

- $(A_0, A_1)_{1,q;K} = (A_0, A_1)_{1,q,(\alpha_0,0)}$ whenever $\alpha_0 < -1/q$ and
- $(A_0, A_1)_{0,q;J} = (A_0, A_1)_{0,q,(\alpha_0,-1)}$ for any $\alpha_0 \in \mathbb{R}$.

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Since $(A_0, A_1)_{0,q,(\alpha_0,\alpha_\infty)} = (A_1, A_0)_{1,q,(\alpha_\infty,\alpha_0)}$, it is sufficient to study the case where $\theta = 1$.

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Case $\theta = 1$. W. D. Evans, B. Opic and L. Pick, J. Inequal. Appl. 7 (2002), 187–269.

$$(A_0, A_1)_{1,q,\mathbb{A}} = \{0\} \text{ unless } \begin{cases} \alpha_0 + 1/q < 0 & \text{if } q < \infty, \\ \alpha_0 \le 0 & \text{if } q = \infty. \end{cases}$$

In these cases, $(A_0, A_1)_{1,q,\mathbb{A}}$ is an interpolation space.

Interpolation of compact operators

D. E. Edmunds, B. Opic, J. Funct. Anal. 266 (2014)

Theorem. Let (R,μ) and (S,ν) be finite measure spaces, let $1 < p_0 < p_1 \le \infty$, $1 < q_0 < q_1 \le \infty$, $1 \le q < \infty$ and $\alpha + 1/q > 0$. Put $\gamma_0 = \alpha + 1/\min(p_0, q)$ and $\gamma_1 = \alpha + 1/\max(p_0, q)$. If T is a linear operator such that

$$T : L_{p_0}(R,\mu) \longrightarrow L_{q_0}(S,\nu) \quad \text{is compact and}$$
$$T : L_{p_1}(R,\mu) \longrightarrow L_{q_1}(S,\nu) \quad \text{is bounded,}$$

then $T : L_{p_0,q}(\log L)_{\gamma_0}(R,\mu) \longrightarrow L_{q_0,q}(\log L)_{\gamma_1}(S,\nu)$ compactly.

F. Cobos, L. M. Fernández-Cabrera, A. Martínez, Math. Nachr. 288 (2015), 167–175.

Let $A_j^o = \overline{A_0 \cap A_1}^{A_j}$

Theorem. Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples with $B_0 \hookrightarrow B_1$. Suppose that $T \in \mathcal{L}(\bar{A}, \bar{B})$ is a linear operator such that

$$\begin{aligned} T : A_0 \longrightarrow B_0 &\quad \text{is bounded and} \\ T : A_1 \longrightarrow B_1 &\quad \text{is compact.} \end{aligned}$$

Let $1 \le q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ such that

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then $T : (A_0^o, A_1^o)_{1,q,\mathbb{A}} \longrightarrow (B_0^o, B_1^o)_{1,q,\mathbb{A}}$ is also compact.

F. Cobos, L. M. Fernández-Cabrera, A. Martínez, Math. Nachr. 288 (2015), 167–175.

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They also showed that one cannot shift the compactness to the first restriction.

Theorem. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$ be such that

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Then for any $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \leq q \leq \infty$ such that $\alpha_0 + 1/q < 0$ if $q < \infty$, or $\alpha_0 \leq 0$ if $q = \infty$, we have that the restriction $T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{1,q,\mathbb{A}}$ is compact.

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Since $(A_0, A_1)_{1,q,(\alpha_0,\alpha_\infty)} = (A_1, A_0)_{0,q,(\alpha_\infty,\alpha_0)}$,

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Corollary. Let (R,μ), (S,ν) be a σ-finite measure space. Take $1 < p_0 < p_1 \le \infty$, $1 < q_0 < q_1 \le \infty$, $1 \le q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$. Let T be a linear operator such that

$$T : L_{p_0}(R) \longrightarrow L_{q_0}(S) \text{ is compact and } T : L_{p_1}(R) \longrightarrow L_{q_1}(S) \text{ is bounded.}$$

Then

$$T : L_{p_0,q}(\log L)_{\mathbb{A}+\frac{1}{\min(p_0,q)}}(R) \longrightarrow L_{q_0,q}(\log L)_{\mathbb{A}+\frac{1}{\max(q_0,q)}}(S)$$

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Corollary. Let (R, μ), (S, ν) be a σ-finite measure space. Take $1 < p_0 < p_1 \leq \infty$, $1 < q_0 < q_1 \leq \infty$, $1 \leq q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$. Let T be a linear operator such that

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is also compact.

For the proof.

→We use the compactness theorem,

→W. D. Evans, B. Opic, Canad. J. Math.(2000): if $r_0 < r_1$ and Ω is σ-finite,

$$L_{r_0,q}(\log L)_{\mathbb{A}+\frac{1}{\min(r_0,q)}}(\Omega) \hookrightarrow (L_{r_0}(\Omega), L_{r_1}(\Omega))_{0,q,\mathbb{A}} \hookrightarrow L_{r_0,q}(\log L)_{\mathbb{A}+\frac{1}{\max(r_0,q)}}(\Omega).$$

If $r_1 < r_0$ and Ω is σ-finite,

$$L_{r_0,q}(\log L)_{\tilde{\mathbb{A}}+\frac{1}{\min(r_0,q)}}(\Omega) \hookrightarrow (L_{r_0}(\Omega), L_{r_1}(\Omega))_{0,q,\mathbb{A}} \hookrightarrow L_{r_0,q}(\log L)_{\tilde{\mathbb{A}}+\frac{1}{\max(r_0,q)}}(\Omega),$$

where $\tilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$.

In particular, if we shift the compactness to the second restriction, the result reads as follows.

Corollary. Let (R,μ), (S,ν) be σ-finite measure spaces. Take $1 \le p_0 < p_1 < \infty$, $1 \le q_0 < q_1 < \infty$, $1 \le q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$. Let T be a linear operator such that

$T : L_{p_0}(R) \longrightarrow L_{q_0}(S)$ is bounded and $T : L_{p_1}(R) \longrightarrow L_{q_1}(S)$ is compact.

Then

$$T : L_{p_1,q}(\log L)_{\mathbb{A}+\frac{1}{\min(p_1,q)}}(R) \longrightarrow L_{q_1,q}(\log L)_{\mathbb{A}+\frac{1}{\max(q_1,q)}}(S)$$

is also compact.

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