Bernoulli Free-boundary Problems
Lecture 2

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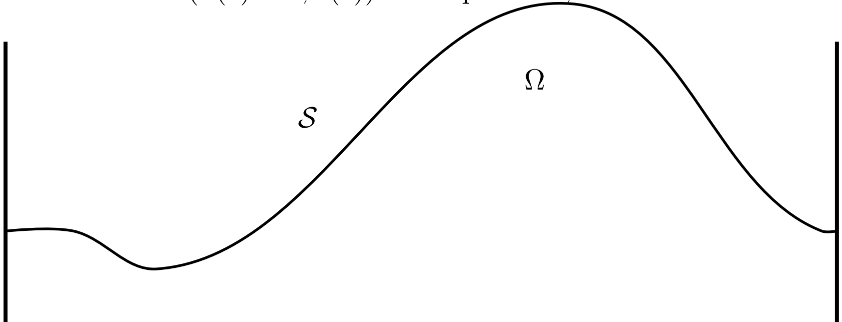
Linköping Lectures 10 – 14 August 2009

Review

Let Ω be the domain below $\mathcal{S}$ in the (X, Y)-plane where

$$\mathcal{S} := \{(u(s), v(s)) \ : \ s \in \mathbb{R}\} \text{ is } 2\pi\text{-periodic,}$$
$$(u, v) \text{ is injective and absolutely continuous,}$$
$$u'(s)^2 + v'(s)^2 > 0 \text{ for almost all } s,$$
$$s \mapsto (u(s) - s, v(s)) \text{ is } 2\pi\text{–periodic,}$$



Bernoulli Free Boundary Problem

Dirichlet problem

$$\psi \in C(\overline{\Omega}) \cap C^2(\Omega),$$
$$\Delta\psi = 0 \text{ in } \Omega, \ \psi \equiv 0 \text{ on } \mathcal{S}, \ \psi \text{ is } 2\pi\text{–periodic in } X,$$
$$\nabla\psi(X,Y) \to (0,1) \text{ as } Y \to -\infty \ \text{ uniformly in } X$$

A Bernoulli free-boundary problem is one of determining $\mathcal{S}$ such that also

$$\boxed{\frac{\partial\psi}{\partial n} = h(Y) \Leftrightarrow |\nabla\psi|^2 = \lambda(Y) \text{ almost everywhere on } \mathcal{S}}$$

where $h = \lambda^2$ is given.

The Neumann derivative can be allowed to depend on the curvature as well as the height of $\mathcal{S}$

Special case: Stokes waves $\lambda(Y) = 1 - 2gY$

Stagnation point: $\lambda(w)(x) = 0$

Conjugation Operator or Hilbert Transform

$\mathcal{C}u$ is defined on $L^2_{2\pi}$ by

$$\mathcal{C}\sin kx = -\cos kx, \quad \mathcal{C}\cos kx = \sin kx, \quad k \in \mathbb{N}, \quad \mathcal{C}1 = 0$$

Note that $w \mapsto \mathcal{C}w'$ is first order and self-adjoint, with Fourier multipliers $|k|$, $k \in \mathbb{Z}$.

- $\mathcal{C}$ is a bounded linear operator on $L^p_{2\pi}$, $1 < p < \infty$ but not in $L^1_{2\pi}$ or $L^\infty_{2\pi}$.
- $\mathcal{H}^{1,1}_{\mathbb{R}}$ be the real *Hardy space* of functions $w \in W^{1,1}_{2\pi}$ with w' in the usual *Hardy space* $\mathcal{H}^1_{\mathbb{R}} := \{u \in L^1_{2\pi} :\ \mathcal{C}u \in L^1_{2\pi}\}$.
- $\mathcal{H}^{1,1}_{\mathbb{R}}$ is a Banach algebra and $\lambda(u) \in \mathcal{H}^{1,1}_{\mathbb{R}}$ when $u \in \mathcal{H}^{1,1}_{\mathbb{R}}$, if λ is Lipschitz continuous.

Let $D \subset \mathbb{C}$ denote the open unit disc. For a holomorphic function $f : D \to \mathbb{C}$, let $f_r(t) = f(re^{it})$ for $t \in \mathbb{R}$ and $r \in (0,1)$.

$$\lim_{r \to 1} \|f_r\|_{L^1_{2\pi}} = \sup_{r\in(0,1)} \|f_r\|_{L^1_{2\pi}} \text{ is well defined}$$

and $u \in \mathcal{H}^1_{\mathbb{R}}$ if and only if $u + i\mathcal{C}u = U^*$ for some $U \in \mathcal{H}^1_{\mathbb{C}}$.

Bernoulli free boundaries yield a solution of

$$\boxed{\lambda(w)\{w'^2 + (1 + \mathcal{C}w')^2\} = 1.} \qquad \textbf{(A)}$$

which is closely related to

$$\boxed{\lambda(w)\big(1 + \mathcal{C}w'\big) + \mathcal{C}\big(\lambda(w)w'\big) = 1,} \qquad \textbf{(B)}$$

Equation (B) is the Euler-Lagrange equation of the functional

$$\boxed{\mathcal{J}(w) = \int_{-\pi}^{\pi} \big\{\Lambda(w)\big(1 + \mathcal{C}w'\big) - w\big\}\, dt, \quad w \in \mathcal{H}^{1,1}_{\mathbb{R}},}$$

Riemann-Hilbert Theory - (**B**) $\Rightarrow$ (**A**) if $\lambda(w) \geq 0$

Theorem

For $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ let $W \in \mathcal{H}^1_{\mathbb{C}}$ be such that $W^* = w' + i(1 + \mathcal{C}w')$. Then the following are equivalent.

(i) *w satisfies* (**B**) *and $\lambda(w) \geq 0$;*

(ii) *w satisfies* (**A**) *and $1/W \in N^+$.*

Moreover

$$w' \in L^2_{2\pi} \Rightarrow \lambda(w)|W^*|^2 = \lambda(w)\{{w'}^2 + (1+\mathcal{C}w')^2 \in L^1_{2\pi} \Rightarrow \lambda(w) \geq 0$$

(B) can be rewritten

$$\begin{aligned} 2\lambda(w)\mathcal{C}w' &= 1 - \lambda(w) + \boxed{\lambda(w)\mathcal{C}w' - \mathcal{C}(\lambda(w)w')} \\ &= 1 - \lambda(w) + \boxed{\mathcal{F}(w)} \end{aligned}$$

The Commutator $\mathcal{F}$

Let

$$G(u)(x,y) = \Lambda(u(y)) - \Lambda(u(x)) - \lambda(u(x))(u(y) - u(x))$$

$$\begin{aligned}
\mathcal{F}(u(x) &= \lambda(u(x))\mathcal{C}u'(x) - \mathcal{C}(\lambda(u)u')(x) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\lambda(u(x)) - \lambda(u(y)))u'(y)}{\tan((x-y)/2)} dy \\
&= \frac{-1}{2\pi}\int_{-\pi}^{\pi} \frac{(\partial/\partial y)G(u)(x,y)}{\tan((x-y)/2)} dy \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{G(u)(x,y)}{\sin^2((x-y)/2)} dy
\end{aligned}$$

If Λ is convex, $G \geq 0$ and $\lambda = \Lambda'$, then $\mathcal{F}(u)(x) \geq 0$ a.e.

When $\lambda(w) = w$ (Stokes Waves,

$$\mathcal{F}(u)(x) = \frac{1}{8\pi}\int_{-\pi}^{\pi} \left(\frac{u(x) - u(y)}{(\sin(x-y)/2)}\right)^2 dy$$

Smoothing of $\mathcal{F}$

Suppose λ is smooth. Then

- $u \in W^{1,2}_{2\pi} \Rightarrow \mathcal{F}(u) \in L^{\infty}_{2\pi}$ and sequentially continuous from the weak $W^{1,2}2\pi$-topology, into $L^p_{2\pi}$, $1 \leq p < \infty$, with the strong L_p-topology.
- If $u \in W^{1,p}_{2\pi}$ for $2 < p < \infty$. Then $\mathcal{F}(u) \in C^{1-\frac{2}{p}}$.
- $u \in C1, \alpha$, $\alpha \in (0,1) \Rightarrow \mathcal{F}(u) \in C^{1,\delta}$, $0 < \delta < \alpha$.

So if $\lambda \neq 0$, a bootstrap gives $u \in C^{\infty}$. Then an independent argument gives that u is real-analytic because λ is real-analytic.

Equation (B) and Bernoulli free boundaries

Theorem

We have observed that every Bernoulli free boundary gives a solution w of

$$\lambda(w)\{w'^2 + (1+\mathcal{C}w')^2\} = 1 \qquad (\mathbf{(A)})$$

In addition it follows that

$$\left.\begin{array}{c} w \text{ satisfies } (\mathbf{B}); \\ \lambda(w) \geq 0; \\ t \mapsto (-(t+\mathcal{C}w(t)), w(t)) \text{ injective on } \mathbb{R}. \end{array}\right\} \qquad (\mathbf{C})$$

Conversely, suppose that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies $(\mathbf{C})$. Let

$$\mathcal{S} = \{(-(t+\mathcal{C}w(t)), w(t)) : \ t \in \mathbb{R}\}$$

and let Ω be the open domain below $\mathcal{S}$. There exists a conformal mapping ω of Ω onto $\mathbb{C}^-$ such that $\mathcal{S}$ gives a solution of a Bernoulli free boundary problem.

Jordan Curves

There is a one-to-one correspondence between solutions of Bernoulli free boundary problems with $|\nabla\psi|$ bounded and solutions $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ of $(\mathbf{C})$

We would like to use the functional $\mathcal{J}$ and its Euler-Lagrange equation $(\mathbf{B})$, without further qualification to study Bernoulli free-boundary problems. Suppose

$$\lambda \geq 0, \quad \log\lambda \text{ is non-constant, concave, and}$$
$$\lambda' \leq 0 \text{ where } \lambda \neq 0 \text{ on } \mathcal{R}(w)$$

Theorem

Suppose this holds and that w is a smooth solution of (B). Then $1 + \mathcal{C}w'(x) > 0$ everywhere and the corresponding Stokes wave is regular and $\mathcal{S}$ is a Jordan curve.

Proof

Suppose $W^* = w' + i(1 + \mathcal{C}w') = |W^*|e^{-i\vartheta}$ where w is a smooth solution of (B). We know that $\log\lambda(w) \in L^1_{2\pi}$ and w is real analytic. It follows from Riemann Hilbert theory that $\vartheta = \mathcal{C}\log\sqrt{\lambda(w)}$. Then ϑ is real analytic.

Since $\log\lambda$ is concave on $\mathcal{R}(w)$ we have proved that

$$\frac{\lambda'(w(t))}{\lambda(w(t))}\mathcal{C}w'(t) - \Big(\mathcal{C}\log\lambda(w)(t)\Big)' \leq 0,$$

equivalently

$$\frac{\lambda'(w(t))}{\lambda(w(t))}\left(\frac{\cos\vartheta}{\sqrt{\lambda(w)}} - 1\right) - 2\vartheta' \leq 0$$

Therefore

$$\vartheta' - \frac{1}{2}\lambda'(w)|W^*|^3 \cos\vartheta > 0$$

Let

$$\vartheta(x^*) = \max_{[-\pi,\pi]} \vartheta$$

and, using translational invariance and periodicity, suppose that

$$\min_{[-\pi,\pi]} \vartheta = \vartheta(x_*), \qquad -\pi < x^* < x_* < \pi$$

Then $\cos\vartheta(x^*) > 0$ and hence

$$\vartheta(x^*) \in ((4k-1)\pi/2, (4k+1)\pi/2) \quad k \in \mathbb{Z}$$

The inequality gives that $\vartheta(x) > (4k-1)\pi/2$ for all $x \in [x^*, x_*]$, for the same k.

Thus $\cos\vartheta$, and equivalently $1 + \mathcal{C}w'$, is everywhere positive.

Result (so far) for less smooth w

Definition

t_0 is called a stagnation point when $\lambda(w(t_0)) = 0$, and solutions with stagnation points are called singular. The set $\mathcal{N}(w)$ of stagnation points is closed.

If

$$\lambda \geq 0, \quad \log \lambda \text{ is non-constant, concave, and}$$
$$\lambda' \leq 0 \text{ where } \lambda \neq 0 \text{ on } \mathcal{R}(w),$$

a solution of $(\mathbf{B})$ defines a *non-self-intersecting* curve $\mathcal{S}$ and $\mathcal{S}$ is a Bernoulli free boundary *provided w has at most countably many stagnation points.*

Stagnation points of w correspond to stagnation points of the free boundary problem $\boxed{\textit{only if}}$ w gives a solution of the free boundary problem

Duality

Recall equation (**B**) in the form

$$\lambda(w)w' + i(-1 + \mathcal{C}(\lambda(w)w')) = \lambda(w)\big(w' - i(1 + \mathcal{C}w')\big)$$

which can be re-written

$$\begin{aligned} -(w' + i(1 + \mathcal{C}w')) &= \frac{1}{\lambda(w)}\Big(- \lambda(w)w' - i(1 + \mathcal{C}(-\lambda(w)w')\Big) \\ &= \frac{1}{\lambda(w)}\Big(v' - i(1 + \mathcal{C}(v')\Big) \end{aligned}$$

where $v = -\lambda(w)$. Suppose that $\lambda(w) \geq 0$ so that (**A**) holds also.

Let $\widetilde{w}(t) = -\int_0^t \lambda(w(x))w'(x)dx$ and $\lambda(w(t))\widetilde{\lambda}(\widetilde{w}(t)) \equiv 1$

Then $\widetilde{w}(t) = -\int_0^t \lambda(w(x))w'(x)dx$ is a solution of (**A**) and (**B**) with $\widetilde{\lambda}$ instead of λ

Dual Stokes Waves

The Stokes wave free boundary conditions are that the harmonic stream function satisfy

$$\boxed{\psi \equiv 0 \text{ and } |\nabla\psi|^2 + 2gy \equiv 1 \text{ on } \mathcal{S}}$$

The dual problem corresponds to to a free-boundary problem for the "dual stream function" $\widetilde{\psi}$:

$$\boxed{\widetilde{\psi} \equiv 0, \qquad (4gy+1)|\nabla\widetilde{\psi}|^4 \equiv 1}$$

at the "dual" free boundary $\widetilde{\mathcal{S}}$

These two apparently distinct Bernoulli problems are equivalent

Self-Duality

An example

It is natural to ask if there are λs such that $\widetilde{\lambda} \equiv \lambda$.
Consider the case $\lambda \in C(\mathbb{R})$, $\lambda(v) > 0, \forall v \in \mathbb{R}$.

Theorem

(i) Suppose $f : [0, +\infty) \to [0, +\infty)$ is continuously differentiable, $f(0) = 0$, $f' > 0$, and $f'(0) = 1$. Let

$$\Lambda(w) = \begin{cases} f(w), & \text{if } w \geq 0, \\ -f^{-1}(-w), & \text{if } w < 0. \end{cases} \tag{1}$$

Then $\lambda = \Lambda'$ is self-dual.

(ii) Conversely, if λ is self-dual, then

$$\Lambda(w) := \int_0^w \lambda(v) dv, \quad w \in \mathbb{R}$$

has the form (1).

Regularity of Solutions of $(\mathbf{B})$

Without hypotheses on sign of $\lambda(w)$ we observe how $\lambda(w) \neq 0$ relates to the regularity of solutions w of $(\mathbf{B})$.

Theorem

When $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ is a solution of $(\mathbf{B})$

- $\log|\lambda(w)| \in L^1_{2\pi}$
- $\lambda(w) > 0$ *on a set of positive measure*
- w *is real-analytic on the open set of full measure* $\lambda(w) \neq 0$

As a corollary, if $\mathcal{S}$, ψ is a Bernoulli free boundary, then

- $\mathcal{S}$ *and* ψ *are real-analytic in a neighbourhood of any point of* $\mathcal{S}$ *that is not a stagnation point,*
- $\nabla\psi$ *is continuous in the closure of* Ω

How zeros of λ affects the smoothness of w

Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of $(\mathbf{B})$. Suppose that $\varrho > 0$ is such that for all $x_0 \in \mathcal{R}(w)$ with $\lambda(x_0) = 0$,

$$|\lambda(x)| \leq \text{constant } |x - x_0|^{\varrho} \text{ for all } x \in \mathcal{R}(w).$$

Let

$$p(\varrho) = \frac{\varrho + 2}{\varrho} \quad \text{and} \quad r(\varrho) = \frac{\varrho + 2}{\varrho + 1}.$$

(a) The following are equivalent:

(i) $w \in W^{1,p(\varrho)}_{2\pi}$ ($w \in W^{1,3}_{2\pi}$ if λ is Lipschitz);
(ii) w is real-analytic on $\mathbb{R}$;
(iii) $\lambda(w) > 0$ on $\mathbb{R}$.

(b) The function w is real-analytic if

$$\lambda(w) \geq 0 \text{ and } -(1+\mathcal{C}w')+iw' = \big|-(1+\mathcal{C}w')+iw'\big|\, e^{i\vartheta},$$

where $\vartheta = \vartheta_1 + \vartheta_2$ with ϑ_1 continuous and $\|\vartheta_2\|_\infty < \pi/(2p(\varrho))$.

($\|\vartheta_2\|_\infty < \pi/6$ if λ is Lipschitz)

(c) If $w \in W^{1,r(\varrho)}_{2\pi}$ then $\lambda(w) \geq 0$ ($w \in W^{1,3/2}_{2\pi}$ if λ is Lipschitz)

(d) If $\varrho = 0$, which amounts to no additional hypothesis since λ is continuous and $\mathcal{R}(w)$ is compact, then $\lambda(w) \geq 0$ if $w \in W^{1,2}_{2\pi}$.

It is not known whether there are solutions of (**B**) which do not satisfy (**A**) for which $\lambda(w)$ changes sign.

There are however solutions w of (**A**) and (**B**) for which $\lambda(w)$ has zeros - the famous Stokes waves

Dimension of the Set of Stagnation Points

It follows from Theorem 6 that $\mathcal{N}(w)$ has measure 0. The following result implies that its dimension is not greater than 2/3 if λ is Lipschitz continuous. Note that the lower Minkowski dimension $\dim_M$, bounds the Hausdorff dimension from above.

Theorem

Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of $(\mathbf{A})$ *and* $(\mathbf{B})$ *where λ is such that*

$$c|x - x_0|^{\varrho} \leq \lambda(x) \leq C|x - x_0|^{\varrho}, \quad c,\, C,\, \varrho > 0,$$

for all x in a neighbourhood of x_0 in $\mathcal{R}(w)$ when $\lambda(x_0) = 0$. Let $q(\varrho) = (\varrho + 2)/2$. Then

$$\dim_M \mathcal{N}(w) \leq 1/q(\varrho).$$

If $w \in W^{1,p}_{2\pi}$, $p > 1$, then

$$\dim_M \mathcal{N}(w) \leq 1-(p/p(\varrho)), \;\; 1 < p < p(\varrho), \;\; \mathcal{N}(w) = \emptyset \text{ when } p \geq p(\varrho).$$

Important Open Question

The hypotheses of this theorem on λ are valid when $\lambda(w) = 1 - 2gw$ for any $g > 0$.

Unfortunately even in that case it is not known whether the requirement that $\mathcal{N}(w)$ be denumerable is necessary.

In fact no examples are known in which $\mathcal{N}(w) \cap [0, 2\pi)$ contains more than one point when $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies (**A**) and (**B**).

Can a solution w of (**A**) and (**B**) have uncountably many stagnation points?