Bernoulli Free-boundary Problems
Lecture 3

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Variational Setting

$\mathcal{C}u$ is defined on $L^2_{2\pi}$ by

$$\mathcal{C}\sin kx = -\cos kx, \quad \mathcal{C}\cos kx = \sin kx, \qquad k \in \mathbb{N}, \qquad \mathcal{C}1 = 0$$

Note that $w \mapsto \mathcal{C}w'$ is first order and self-adjoint, with Fourier multipliers $|k|$, $k \in \mathbb{Z}$.

- $\mathcal{H}^{1,1}_{\mathbb{R}}$ be the real *Hardy space* of functions $w \in W^{1,1}_{2\pi}$ with w' in the usual *Hardy space* $\mathcal{H}^1_{\mathbb{R}} := \{u \in L^1_{2\pi} : \mathcal{C}u \in L^1_{2\pi}\}$.
- $\mathcal{H}^{1,1}_{\mathbb{R}}$ is a Banach algebra and $\lambda(u) \in \mathcal{H}^{1,1}_{\mathbb{R}}$ when $u \in \mathcal{H}^{1,1}_{\mathbb{R}}$, if λ is Lipschitz continuous.

$$\boxed{\lambda(w)\big(1 + \mathcal{C}w'\big) + \mathcal{C}\big(\lambda(w)w'\big) = 1,} \qquad \textbf{(B)}$$

is the Euler-Lagrange equation of the functional

$$\boxed{\mathcal{J}(w) = \int_{-\pi}^{\pi} \big\{\Lambda(w)\big(1 + \mathcal{C}w'\big) - w\big\}\, dt, \quad w \in \mathcal{H}^{1,1}_{\mathbb{R}},}$$

Basic Requirements of a Useful Theory

Recall from Lecture 2 that if solutions of (B) are to give Bernoulli free boundaries then we need to know that $\lambda(w) \geq 0$ and that

$$\{(-t - \mathcal{C}w(t), w(t)) : t \in \mathbb{R}\}$$

defines a 2π-periodic *Jordan* curve.

For λ Lipschitz, $\lambda(w) \geq 0$ if $w \in W^{1,3/2}_{2\pi}$ and if

$$\lambda \geq 0, \quad \log\lambda \text{ is non-constant, concave, and} \\ \lambda' \leq 0 \text{ where } \lambda \neq 0 \text{ on } \mathcal{R}(w), \tag{$\dagger$}$$

a solution of $(\mathbf{B})$ defines a *non-self-intersecting* curve $\mathcal{S}$

Therefore, for λ Lipschitz, we should (until we know more) assume that λ satisfies $(\dagger)$ and we need solutions in $W^{1,3/2}_{2\pi}$

Morse Indices

Despite the simple and attractive form of $\mathcal{J}$, the global variational theory remains largely unexplored – see later

First we develop a Morse index theory in the Hilbert space $W^{1,2}_{2\pi}$ for non-singular (no stagnation points) critical points – such solutions are regular and satisfy both (A) and (B)

The Morse index of a critical point w of $\mathcal{J}$ is the number of negative eigenvalues of $D^2\mathcal{J}(w)$:

$$D^2\mathcal{J}(w)\,u = \lambda'(w)(1+\mathcal{C}w')u + \lambda(w)\mathcal{C}u' + \mathcal{C}\big(\lambda'(w)w'u + \lambda(w)u'\big)$$

Clearly $D^2\mathcal{J}(w) : W^{1,2}_{2\pi} \to L^2_{2\pi}$ and, for $u \in W^{1,2}_{2\pi}$,

$$Q_w(u) = \langle D^2\mathcal{J}(w)u, u\rangle_{L^2_{2\pi}} = \int_{-\pi}^{\pi} \big\{\lambda'(w)(1+\mathcal{C}w')u^2 + 2\lambda(w)u\mathcal{C}u'\big\}dt$$

defines a quadratic form on $W^{1,2}_{2\pi}$. The Morse index of w is

$$\mathcal{M}(w) = \sup\,\big\{\dim E \,:\, Q_w(u) < 0,\ u \in E\setminus\{0\}\big\}$$

where E denotes a linear subspace of $W^{1,2}_{2\pi}$.

Plotnikov's Transformation

Plotnikov showed that the Morse index is given by the much more convenient formula

$$\mathcal{M}(w) = \sup \left\{ \dim E \,:\, \mathcal{Q}_w(u) < 0,\ u \in E \setminus \{0\} \right\},$$

where E denotes a linear subspace of $W^{1,2}_{2\pi}$,

$$\boxed{\mathcal{Q}_w(u) := \int_{-\pi}^{\pi} \{u\mathcal{C}u' + qu^2\} dt}$$

and, with $\vartheta = \mathcal{C}\big(\log \sqrt{\lambda(w)}\big)$, the potential

$$q = -\vartheta' + \frac{\lambda'(w)}{2\lambda(w)}(1 + \mathcal{C}w') = -\vartheta' + \frac{1}{2}\lambda'(w)\lambda(w)^{-3/2} \cos\vartheta.$$

Like Magic: the convolution formula leads to the conclusion that Plotnikov's potential q is negative

Theorem

Suppose that w is a non-singular solution of and let $\widetilde{w}$ denote the solution of the dual problem. Then the Morse indices of w and $\widetilde{w}$ are equal.

When λ is globally real-analytic, the operator $D^2\mathcal{J}(w)$ is well-defined at any $W^{1,2}_{2\pi}$-solution w irrespective of the smoothness of w. But the Morse index may be infinite if w has stagnation points.

The next result says that if the Morse indices of a set of non-singular solutions are bounded, then none of them is close to being singular. This is true is for families of systems with possibly different nonlinearities $\lambda_k(w)$.

Theorem

Suppose that a sequence $\{w_k\}$ of non-singular solutions of systems with $\lambda = \lambda_k$ in a certain admissible class and that $\{\mathcal{M}(w_k)\}$ is bounded. Then, for some $\alpha > 0$,
$\lambda_k(w_k(t)) \geq \alpha, \quad k \in \mathbb{N}.$

To Focus on Difficulties – Consider Stokes Waves

There is no essential difference in the more general setting

Then $\lambda(w) = 1 - 2\lambda w$, satisfies (†) and equation (B) has the form

$$\mathcal{C}w' = \lambda(w + \mathcal{C}w' + w\mathcal{C}w' + \mathcal{C}(ww')), \quad w \in \mathcal{H}^{1,1}_{\mathbb{R}} \qquad \textbf{(S)}$$

or, in terms of the commutator $\mathcal{F}$,

$$(1 - 2\lambda w)\mathcal{C}w' = \lambda(w - \mathcal{F}(w))$$

where

$$\mathcal{F}(w)(x) = (w\mathcal{C}w' - \mathcal{C}(ww'))(x)$$

$$= \frac{1}{8\pi}\int_{-\pi}^{\pi}\left(\frac{u(x) - u(y)}{\sin(x-y)/2)}\right)^2 dy \geq 0$$

The Corresponding Functional

$\lambda(w) = 1 - 2\lambda w$ and $\Lambda(w) = w - \lambda w^2$

For $w \in \mathcal{H}_{\mathbb{R}}^{1,1}$,

$$\mathcal{J}(w) = \int_{-\pi}^{\pi} \left\{\Lambda(w)\big(1 + \mathcal{C}w'\big) - w\right\} dt$$
$$= \int_{-\pi}^{\pi} w\mathcal{C}w'\, dt - \lambda \int_{-\pi}^{\pi} w^2\, dt - \lambda \int_{-\pi}^{\pi} w^2\mathcal{C}w'\, dt$$

Note that if $w = \sum_{k\in\mathbb{Z}} w_k e^{ikt}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} w\mathcal{C}w'\, dt = \sum_{k\in\mathbb{Z}} |k||w_k|^2 = \Big(\|w\|^2_{H^{1/2}_{2\pi}} - \|w\|^2_{L^2_{2\pi}}\Big),$$

But $H^{1/2}_{2\pi} \not\subset L^\infty_{2\pi}$ and $w^2\mathcal{C}w' \notin L^1_{2\pi}$ for $w \in H^{1/2}_{2\pi}$.
If we work in $H^{1/2+\varepsilon}_{2\pi}$ the functional does not give bounds needed for the Palais-Smale condition

Coping by Penalization–Regularization

The penalization and regularization strategy which Mark Groves used to great effect for $3D$ solitary waves can be implemented here also.

But the results are disappointing:

$$\text{for } \lambda \in (0.99, 1) \text{ there exists a non-zero solution of } (\mathbf{S})$$

The the use of regularization and penalization, and the need to prove that the solution so found is non-trivial, curtails the effectiveness of this potentially global variational method to yield solutions close to bifurcation points.

Ignoring the variational setting, consider equation (B) as an operator equation $G(\lambda, w) = 0$ with

$$G(\lambda, w) = (1 - 2\lambda w)\mathcal{C}w' - \lambda(w - \mathcal{F}(w))$$

where

$$\mathcal{F}(w)(x) = (w\mathcal{C}w' - \mathcal{C}(ww'))(x) = \frac{1}{8\pi} \int_{-\pi}^{\pi} \left(\frac{u(x) - u(y)}{\sin(x-y)/2} \right)^2 dy \geq 0$$

Local Bifurcation Theory

a la Implicit Function Theorem

$w = 0$ is a solution (the trivial solution) for all λ.

Linearized about zero, the problem is $\mathcal{C}w' = \lambda w$ and the solutions are $\lambda = k$, $k \in \mathbb{N} \cup 0$, and $w \in$ span$\{\sin kt, \cos kt\}$

For simplicity, we seek only symmetric waves (i.e. even w)

Standard bifurcation theory at $\lambda = 1$ leads to the existence locally of a real-analytic curve of even solutions

$B = \{(\lambda_\epsilon, w_\epsilon), |\epsilon| \leq \epsilon_0\} \subset \mathbb{R} \times C^N_{2\pi}$, for any N

with $\lambda_0 = 1$, $w_0 = 0$ and

$w_{-\epsilon}(t) = w_\epsilon(t + \pi)$ and $\lambda_{-\epsilon} = \lambda_\epsilon < 1$ if $\epsilon \neq 0$

Global Bifurcation Theory

a la Leray-Schauder Degree or *Real-Analytic Variety Theory*

Moreover, using degree theory, it can be shown that there is a continuum $\mathcal{B}$ of such solutions in $\mathbb{R} \times C_{2\pi}^{N}$ with the following properties:

- $B \subset \mathcal{B}$;
- $0 < a \leq \lambda \leq b < \infty$ for all $(\lambda, w) \in \mathcal{B}$;
- w is even and monotone on $[0, \pi]$ if $(\lambda, w) \in \mathcal{B}$
- There is a sequence $(\lambda_k, w_k) \in \mathcal{B}$ such that $1 - 2\lambda_k w_k(0) \to 0$

Indeed, because our equation involves only real-analytic nonlinear operators, B has a unique global extension as a one-dimensional curve with a real-analytic parametrization at each point, self-intersections and encounters with other manifolds of solutions notwithstanding

Consequently $\mathcal{M}(w_k) \to \infty$ as $k \to \infty$ and there are solutions of arbitrarily large Morse index on the continuum

Bifurcations and Secondary Bifurcations on $\mathcal{B}$

To finish, we give a brief sketch of how real-analytic bifurcation theory interacts with the variational structure to conclude the existence of multiple secondary bifurcation points on the global branch $\mathcal{B}$.

The abstract theory upon which these conclusions are based considerations of real-analyticity

and it t is worth emphasizing that the existence of a path, not just a connected set, of solutions is essential. This is where the real-analyticity comes in

Definition

(λ_0, y_0) is a bifurcation point for an equation $G(\lambda, y) = 0$ if there are two sequences $\{(\lambda_k, \widehat{y}_k)\}$, $\{(\lambda_k, \widetilde{y}_k)\}$ of solutions of $G(\lambda, y) = 0$ with $\widetilde{y}_k \neq \widetilde{y}_k$ for all k (same λ_k for both) converging to (λ_0, y_0) in $\mathbb{R} \times Y$. $\square$

Suppose that Y is dense in a Hilbert space $(X, \langle\cdot,\cdot\rangle)$ and that $G(\lambda,\cdot)$ is the gradient of a C^2-functional $g(\lambda,\cdot)$ with $G(\lambda, y) = 0$ and *the linearization* $\partial_y G[(\lambda, y)] - \mu\iota : Y \to X$ a homeomorphism except for μ in a discrete set $\mathbb{S}(\lambda, y)$. For $\mu \in \mathbb{S}(\lambda, y)$ suppose that $\big(\mu\iota - \partial_y G[(\lambda, y)]\big)$ is a Fredholm operator of index zero.

this is guaranteed for $\mathcal{B}$ by the real analytic bifurcation theory

Lemma

Suppose that $\mathcal{U} \subset (0,\infty) \times Y$ *is an open set,* $G : \mathcal{U} \to X$ *is* C^2 *and such that* $M(\lambda, y)$ *is well-defined for every* $(\lambda, y) \in \mathcal{U}$ *with* $G(\lambda, y) = 0$. *Suppose also that for compact sets of solutions in* $\mathcal{U}$*, the sets* $\mathbb{S}(\lambda, y)$ *are uniformly bounded below. Let* $S := \{(\lambda(s), y(s)) : s \in (-\epsilon, \epsilon)\} \subset \mathcal{U}$ *be a continuous curve of solutions to* $G(\lambda, y) = 0$ *such that* $0 \notin \mathbb{S}(\lambda(s), y(s))$ *for all* $s \in (-\epsilon, \epsilon) \setminus \{0\}$ *and that*

$$\boxed{\lim_{s \nearrow 0} M(\lambda(s), y(s)) \neq \lim_{s \searrow 0} M(\lambda(s), y(s))}$$

Then $(\lambda(0), y(0))$ *is a bifurcation point.*

In the topological version of global bifurcation theory it is difficult, and in general not always possible, to be sure of the existence of such a path upon which secondary bifurcation points can be identified.

The following result is now immediate from the existence of a path of solutions given by the analytic global bifurcation theory

Lemma

There is an infinite discrete set Σ of values of points on $\mathcal{B}$ which are a bifurcation points for the Stokes wave equation.

It is not known whether they are turning points or secondary bifurcation points – the numerical evidence points to the turning points but this is completely open mathematically

Strong numerical evidence suggests that in the physical domain the curve $\mathcal{B}$ gives a *maximal connected set* of Stokes waves of the *fundamental period* 2π, and that no Stokes waves of period 2π bifurcate from it and that it does it self-intersect.

Scaling

It is easy to see that if (λ, w) is a solution of the Stokes wave equation (B), then so is $(k\lambda, k^{-1}w(kt))$.

So $\mathcal{B}$ has a scaled copy $\mathcal{B}_k$ that bifurcates from $(k, 0)$.

Each of these branches has solutions of minimal period $2\pi/k$ and they can be scaled back to $\mathcal{B}$

A question asked by Levi-Civita is this: can every solution bifurcating from $(k, 0)$ be scaled back to a point on the branch bifurcating fom (1,0)?

The answer, from this variational theory, is NO!

Because the Morse index tends to infinity along $\mathcal{B}$, there exist secondary bifurcation points on $\mathcal{B}_k$ that are not copies of behaviour on $\mathcal{B}$

Hence *the formation of singular waves with stagnation points is the cause, mathematically, of secondary* sub-harmonic *bifurcations on $\mathcal{B}_k$*

When interpreted in the physical domain, they correspond to *period-multiplying* bifurcations of Stokes waves. The physical waves which bifurcate have minimal period $2k\pi$ and there are infinitely many period-multiplying bifurcation points for Stokes waves in the physical domain

NEVERTHELESS THERE IS ESSENTIALLY NO VARIATIONAL THEORY OF THE OCCURRENCE OF THESE PHENOMENA

Collaborators

the Real Mathematicians

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- Eugen Varvaruca (Imperial College London)