Analysis of a Bäcklund transformation

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Let f be a smooth Young function; let x and y denote rectangular coordinates in Euclidean plane $\mathbb{R}^2$, and let n be a smooth positive function of x and y. In the present paper we outline a tentative theory and an application of the Bäcklund transformation that is attached to the following equation

$$\nabla v = \frac{n}{|\nabla u|} f'\!\left(\frac{|\nabla u|}{n}\right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla u \tag{1}$$

and consists in suitably stretching or shrinking the gradient of a function u of x and y, plus rotating the same gradient by ninety degrees.

Equation (1) amounts to coupling the following system

$$\begin{gathered} |\,\nabla u\,| = n \cdot f'\!\left(\frac{|\nabla u|}{n}\right) \\ (\nabla u, \nabla v) = 0 \end{gathered} \tag{2}$$

and the following condition

$$\frac{\partial(u,v)}{\partial(x,y)} \geq 0.$$

In the case where $0 \leq \epsilon < 1$ and

$$f(\rho) = \int_0^\rho t \left(\frac{1+t^2}{\epsilon + t^2}\right)^{\frac{1}{2(1-\epsilon)}} dt,$$

(2) is dictated by investigations about viscosity solutions to the following system

$$\begin{gathered} |\,\nabla u\,|^2 - |\,\nabla v\,|^2 = n^2 \\ (\nabla u, \nabla v) = 0, \end{gathered} \tag{3}$$

which arises when an asymptotic expansion of solutions U to the Helmholtz equation

$$\Delta U + \nu^2 n^2 U = 0 \tag{4}$$

is sought in the following form

$$U \sim A \cdot \mathrm{e}^{-\nu u} \cdot \cos(\nu v) + B \cdot \mathrm{e}^{-\nu u} \cdot \sin(\nu v) \tag{5}$$

under the hypotheses that A, B, u, v do not depend on ν, and $\nu \to \infty$.

Recall that (4) is a prototype of those partial differential equations that ensue from the Maxwell system and model the interaction between an isotropic, non-conducting, non-dissipative medium and a monochromatic electromagnetic field in absence of electric charges. In such a context, n and ν stand for the refractive index of the medium and the wave number, respectively. Expansion (5) is the starting point of a theory, which has been proposed by L. Felsen and coworkers and nicknamed Evanescent Wave Tracking (EWT). EWT both includes geometrical optics, and is credited to describe some optical phenomena (such as the development of fast decaying waves past a caustic) that are excluded from geometrical optics.