Geometry of black hole thermodynamics: Energy VS Entropy representation

Narit Pidokrajt

Department of Physics, Stockholm University
narit@fysik.su.se

GRSweden 2011

Linköping
20 May 2011
1. GRSweden day, Fysikum Stockholm University, February, 2003
2. GRSweden meeting, Matematiska institutionen, Linköpings universitet, June 3, 2003
3. GRSweden meeting, Matematiska institutionen, Linköpings universitet, April 23, 2004
4. Geometry and Relativity Meeting, Matematiska institutionen, Linköpings universitet, Feb 18, 2005
5. GRSweden meeting, Matematiska institutionen, Linköpings universitet, June 14-15, 2006
7. XXIX Spanish Relativity Meeting (E.R.E 2006), Palma de Mallorca, Spain, September 4-8, 2006
9. GRSweden meeting, Matematiska institutionen, Linköpings universitet, 26-27 April 2007
10. Spanish Relativity Meeting (ERE2007), 10-14 September 2007, Tenerife, Spain
12. GRSweden meeting in Linköpings, 4 August 2009
13. ERE2009 in Bilbao, Spain, 7-11 September 2009
| 1. | Geometry of black hole thermodynamics.  
      Published in Gen.Rel.Grav. 35 (2003) 1733  
      e-Print: gr-qc/0304015 |
|---|---|
| 2. | Geometry of higher-dimensional black hole thermodynamics.  
      Published in Phys.Rev. D73 (2006) 024017  
      e-Print: hep-th/0510139 |
| 3. | Flat information geometries in black hole thermodynamics.  
      Published in Gen.Rel.Grav. 38 (2006) 1305-1315  
      e-Print: gr-qc/0601119 |
| 4. | Ruppeiner theory of black hole thermodynamics.  
    - **Jan E Aman (Stockholm U.). James Bedford (Queen Mary, U. of London & CERN), Daniel Grumiller (MIT, LNS), Narit Pidokrajt** |

"SPIRES topcites for this research field"
Special thanks to

- Kvant- och Fältteori, Fysikum, Stockholms universitet
- KVA: The Royal Swedish Academy of Sciences (travel grant, FOA10V-116)
- Helge Ax:son Johnson Stiftelse (research grant 2010-2011)
- Physics Department, Szegedi Tudományegyetem (University of Szeged)
- IGAIA3 (The third conference on Information Geometry and its Applications 2010), Leipzig, Germany

A book to read this summer about Roy Kerr and his achievements:

Cracking the Einstein Code: RELATIVITY AND THE BIRTH OF BLACK HOLE PHYSICS
University of Chicago Press
Who is who?

Collaborators:

1. Jan E. Åman (Stockholm U.)
2. James Bedford (Imperial College London)
3. Ingemar Bengtsson (Stockholm U.)
4. Laszlo A. Gergely (Szeged U.)
5. Daniel Grumiller (TU Wien)
6. John Ward (U. of Victoria)
7. Sergei Winitzki (NEXT Munich, IT business)

Consultants:

1. Roberto Emparan (Barcelona U.)
2. Gary Gibbons (Cambridge U.)
3. Des Johnston (Hariott-Watt U.)
Plan of this talk:

1. Information geometry and applications
2. Black hole thermodynamics & problems that come with it
3. Some results (selected)
   - Kerr-Newman family and AdS black holes
   - Myers-Perry black holes and black objects
   - Energy, Entropy or Free energy?
4. Ongoing projects
5. Outlook
Information Geometry and its Applications

Registration and further information:
www.mis.mpg.de/calendar/conferences/2010/infgeo.html

Scientific Organizers
Nihat Ay
Max Planck Institute for Mathematics in the Sciences
Information Theory of Cognitive Systems Group
Germany

Paolo Gibilisco
Università degli Studi di Roma "Tor Vergata"
Facoltà di Economia
Italy

František Matúš
Academy of Sciences of the Czech Republic
Institute of Information Theory and Automation
Czech Republic

Scientific Committee
Shun-ichi Amari (RIKEN, Japan)
Pierre-Yves Bourguignon (Max Planck Institute for Mathematics in the Sciences, Germany)
Dorje Brody (Imperial College London, United Kingdom)
Frank Critchley (The Open University, United Kingdom)
Imre Csiszár (Hungarian Academy of Sciences, Hungary)
Philip Dawid (University of Cambridge, United Kingdom)
Christopher Dodson (University of Manchester, United Kingdom)
Shinto Eguchi (Institute of Statistical Mathematics, Japan)
Christopher A. Fuchs (Perimeter Institute for Theoretical Physics, Canada)
Akihiko Fujisawa (Dazaifu University, Japan)
Koji Fukuyama (Kyoto University, Japan)
Koji Fukushima (The Institute of Statistical Mathematics, Japan)

Invited Speakers
Shun-ichi Amari (RIKEN, Japan)
Pierre-Yves Bourguignon (Max Planck Institute for Mathematics in the Sciences, Germany)
Dorje Brody (Imperial College London, United Kingdom)
Frank Critchley (The Open University, United Kingdom)
Imre Csiszár (Hungarian Academy of Sciences, Hungary)
Philip Dawid (University of Cambridge, United Kingdom)
Christopher Dodson (University of Manchester, United Kingdom)
Shinto Eguchi (Institute of Statistical Mathematics, Japan)
Christopher A. Fuchs (Perimeter Institute for Theoretical Physics, Canada)
Akihiko Fujisawa (Dazaifu University, Japan)
Koji Fukuyama (Kyoto University, Japan)
Koji Fukushima (The Institute of Statistical Mathematics, Japan)

Administrative Contact
Antje Vandenberg
Max Planck Institute for Mathematics in the Sciences
Germany
Information Geometry (IG)

- IG is the study of probability and information by way of differential geometry.
- Treat structures in probability theory, information theory and statistics as structures in differential geometry by regarding a space of probability distributions as a differentiable manifold endowed with a Riemannian metric.
- We focus on the Fisher information (to come)
Applications of IG

1. Neural networks and cognitive systems: Brain science
2. Computational and systems biology: Bioinformatics
3. Mathematical finance
4. Statistical mechanics
5. Quantum information and quantum statistics
6. Gravitational physics
7. Statistical modelling
8. Unique applications: US and Greek Naval research teams work on IG methods for wave predictions for the North Atlantic ocean.
For our purpose we study metrics that are defined (in some preferred affine coordinate system) by

\[ g_{ab} = \partial_a \partial_b \psi \]  

(1)

The potential \( \psi \) can be any reasonable function. One example is

\[ \psi = \sum_{i=1}^{N} x^i \ln x^i, \quad x^i > 0 \]  

(2)

This potential is a negative of Shannon entropy. The Hessian of \( \psi \) is the Fisher information matrix. With some affined coordinate system it is regarded as a metric. If \( \psi \) is free energy then this metric is called Fisher-Rao metric. The derived geodesic distance, known as Rao distance

\[ ds^2 = \frac{\partial^2 F}{\partial X^a \partial X^b} dX^a dX^b \]  

(3)

And \( ds \) provides a measure of difference between two probability distributions.

For more detailed information, see Geometry of Quantum States, printed by Cambridge University Press. Authors: Ingemar Bengtsson and K. Życzkowski.
In late 1970s, thermodynamic geometry program was started. If the potential \( \psi \) is a negative entropy, namely

\[
\psi = -S(X, N^i)
\]  

(4)

the corresponding metric \( g_{ab} = -\partial_a \partial_b S \) is known as the Ruppeiner metric. For

\[
\psi = M(S, N^i)
\]  

(5)

it is called Weinhold metric. The parameters \( N^i \) are mechanically conserved (extensive thermodynamic variables).

The two metrics are related by a conformal relation:

\[
ds^2_W = T ds^2_R , \quad T \equiv \left( \frac{\partial M}{\partial S} \right)
\]  

(6)

The Ruppeiner curvature scalar is physically meaningful. For ideal gas \( \mathcal{R}_R = 0 \). For a system with interacting statistics such as van der Waals gas, the Ising model, \( \mathcal{R}_R \) diverges at the critical point. Metric signature is related to sign of specific heat (stability...?).

An intriguing question is: What does \( \mathcal{R}_R \) mean for black holes? . The answer is: It is a very interesting question, indeed.
Pre-Hawking black holes

Schwarzschild BH is described solely by $M$;

\[ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 dS^2 \]  \hspace{1cm} (7)

It is a member of the Kerr-Newman black hole family characterized by $(M, J, Q)$. 
One of the most puzzling aspects of the BH is that it is a thermodynamical system with entropy (Bekenstein): $S \sim A$.

Hawking: BHs ain’t black! There is a thermal radiation with a black body spectrum emitted by black holes due to quantum effects. BH temperature: $T \propto \kappa$ and Hawking fixed the proportionality constants for $T$ and $S$

$$T_{BH} = \frac{\hbar \kappa}{2\pi c k_B} \quad S_{BH} = \frac{k_B A}{4\ell_P^2}$$

(8)

where $\ell_P$ is the Planck length, $\ell_P = \sqrt{G\hbar/c^3}$.

Hawking will be in Uppsala on 2 July 2011 to give a public lecture.

NB: Extremal BH has $T = 0$. Due to degenerate horizons. In some sense: mass is balanced with charge. For BHs to exist physically mass should be larger than charge (in normalized units).
Today’s black holes

A black hole which forms, and then completely evaporates away and we are back to the Minkowski space (but not without any problem!). Information falling into the black hole will hit the singularity.
The four laws of black hole mechanics analogous to thermodynamic laws:

▶ **Zeroth law:** Surface gravity $\kappa$ is constant on the BH event horizon $\leftrightarrow T_{BH}$ is constant throughout a body in thermal equilibrium and $T_{BH} = \frac{\kappa}{2\pi}$

▶ **First law:**

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$$

(9)

Because $S = \frac{A}{4}$ thus

$$dM = T_{BH} dS + \Omega dJ + \Phi dQ$$

(10)

with $T_H = \frac{\kappa}{2\pi}$.

By definition: $T = \partial_S M, \Omega = \partial_J M$ and $\Phi = \partial_Q M$.

▶ **Second law:** $dA \geq 0 \leftrightarrow dS \geq 0$

▶ **Third law:** $T = 0$ cannot be reached (weak version) $\leftrightarrow$ the BH extremal states are inaccessible. Exception: Schwarzschild BH with $M \to \infty$.

It is OK that $T \Rightarrow 0$ as $S \Rightarrow 0$ (strong version), then we have extremal BH with $T = 0$ but $S \neq 0$, some violation there. So the third law of BH thermodynamics is unsettling.
Gravity is long-range interaction $\Rightarrow$ gravitating bodies including BHs are nonextensive systems.

\[ S(\lambda U, \lambda V) \neq \lambda S(U, V) \]  
with $\lambda$ a scaling variable

- Most BHs have negative specific heat (the more it radiates the hotter it becomes)

- Astrophysical observations show that most BHs in nature are Kerr BHs (in fact they spin ultrafast!), i.e. most BHs are near-extremal BHs.
BH thermodynamics IV

BH thermodynamics shows that there is a very deep and fundamental relationship between gravitation, thermodynamics, and quantum theory but there are some big problems e.g.

- Information loss problem
  Hawking: Information gets lost! Gravity and Quantum Mechanics are just plain incompatible. Susskind: Hawking was wrong! Black holes are not information-erasers but information-scramblers!

- Origin of BH entropy ⇔ BH statistical mechanics
Thermodynamics is a well-defined macroscopic theory and its microscopic description is the quantum statistical mechanics.

\[ S = k_B \ln \Omega \quad \Omega = \text{a number of microstates} \]

As of today we do not have a fully consistent microscopic description of black holes. Since the microscopic description seems to require a quantum theory of gravity ⇒ study of BH entropy is the study of some aspects of quantum gravity.
Edward Witten—probably the most brilliant theoretical physicist of our time, and Narit at KVA in Stockholm. Witten has not yet been able to erect the full-fledged quantum theory of gravity.
\[ S = 2M^2 - Q^2 + 2M^2 \sqrt{1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4}} \]  

(12)

The Ruppeiner metric of the Kerr black hole \((Q = 0)\) is

\[
ds^2_R = \frac{2}{\left(1 - \frac{J^2}{M^4}\right)} \left\{ -2 \left[ \left(1 - \frac{J^2}{M^4}\right)^{3/2} + 1 - \frac{3J^2}{M^4} \right] dM^2 - \frac{4J}{M^3} dMdJ + \frac{dJ^2}{M^2} \right\} 
\]

(13)

The diagonalized metric reads (using \(v = \frac{J}{M^2}; -1 \leq v \leq 1\).)

\[
ds^2_R = -2 \left(1 + \frac{2}{\sqrt{1 - v^2}} \right) dM^2 + \frac{2M^2}{(1 - v^2)^{3/2}} dv^2 
\]

(14)

The Ruppeiner curvature scalar is

\[
\mathcal{R} = \frac{1}{4M^2} \frac{\sqrt{1 - \frac{J^2}{M^4} - 2}}{\sqrt{1 - \frac{J^2}{M^4}}}. 
\]

(15)

This curvature scalar diverges in the extremal limit \((J = M^2)\).
\[ S(M, Q) = M^2 - Q^2 + M^2 \sqrt{1 - \frac{Q^2}{M^2}} \]  

(16)

\[
\begin{align*}
\left( dS^2_W \right) &= \frac{1}{8S^{\frac{3}{2}}} \left[ - \left( 1 - \frac{3Q^2}{S} \right) dS^2 - 8QdSdQ + 8SdQ^2 \right] \\
\end{align*}
\]

(17)

with \( u = \frac{Q}{\sqrt{S}} \); \(-1 \leq u \leq 1\)

\[
\left( ds^2_R \right) = -\frac{dS^2}{2S} + 4S \frac{du^2}{1 - u^2} 
\]

(18)

This is a flat metric and can be embedded in Minkowskian-like cone
Reissner-Nordström AdS

\[ M = \frac{\sqrt{S}}{2} \left( 1 + \frac{S}{l^2} + \frac{Q^2}{S} \right) ; \quad \frac{Q^2}{S} = 1 + \frac{3S}{l^2}, \quad (19) \]

which is consistent with the extremal limit of the ordinary RN black hole when the cosmological constant is switched off (\( \Lambda = 0 \) i.e. \( l \to \infty \)).

\[ ds^2_W = \frac{1}{8S^{3/2}} \left[ - \left( 1 - \frac{3S}{l^2} - \frac{3Q^2}{S} \right) dS^2 - 8QdSdQ + 8SdQ^2 \right] \quad (20) \]

\[ ds^2_R = \frac{1}{1 + \frac{3\tau^2}{2l^2} - u^2} \left[ - \left( 1 - \frac{3\tau^2}{2l^2} - u^2 \right) d\tau^2 + 2\tau^2 du^2 \right] \quad (21) \]

We can observe that the metric above changes its properties as the signature of the metric changes. It is related to the Hawking-Page phase transition.

\[ \mathcal{R} = \frac{9}{l^2} \frac{\left( \frac{3S}{l^2} + \frac{Q^2}{S} \right) \left( 1 - \frac{S}{l^2} - \frac{Q^2}{S} \right)}{\left( 1 - \frac{3S}{l^2} - \frac{Q^2}{S} \right)^2 \left( 1 + \frac{3S}{l^2} - \frac{Q^2}{S} \right)} \quad (22) \]

Curvature scalar diverges both in the extremal limit and along the curve where the metric changes signature.
Einstein black holes in hyperspace

Event horizon is obtained through:

\[ r_+^2 + a^2 - \frac{\mu}{r_+^{d-5}} = 0. \] (23)

but not analytic. Jan Åman was first to obtain Kerr black hole mass (as a function of thermodynamic parameters) in \( D \) dimension

\[ M = \frac{D - 2}{4} S^{(D-3)/2} \left( 1 + \frac{4J^2}{S^2} \right)^{1/(D-2)} \] (24)

\[ \mathcal{R}_R = -\frac{1}{S} \frac{1 - 12 \frac{d - 5}{d - 3} \frac{J^2}{S^2}}{\left( 1 - 4 \frac{d - 5}{d - 3} \frac{J^2}{S^2} \right) \left( 1 + 4 \frac{d - 5}{d - 3} \frac{J^2}{S^2} \right)}. \] (25)

For \( d \geq 6 \) we have a curvature blow-up but not in the limit of extremal black hole, rather at

\[ 4J^2 = \frac{d - 3}{d - 5} S^2. \] (26)

Emparan and Myers (2003) suggest that the Kerr black hole becomes unstable and changes its behavior to be like a black membrane approximately at this point. So \( \mathcal{R}_R \) predicts the onset of ultraspinning instability of Myers-Perry black holes.
Black hole phase transitions
Gravitational Wave emissions and information geometry. How would these fit together?
We only worry about the plunge/merger case i.e. two equal-mass BHs merge to become a single BH. This is the case of strongest GW emission. BH merger is by far the most energetic event in the post big-bang era.

The upper limit for Gravitational Wave emission is 29 % based on Hawking’s thermodynamic argument [Phys. Rev. Lett. 26, 1344 (1971)]. Based on no-hair theorem.

\[ \eta = 1 - \frac{m_f}{m_1 + m_2} \] (27)

But the upper limit from numerical GR gives between 4-11 % [e.g. Pretorius 0710.1338]

We want sophisticated thermodynamic argument giving numbers much close to that comes out from numerical works.

Information geometry = the way to improve this? L. Gergely and I are trying anyway.
Ruppeiner geometry is physically suggestive whereas its counterpart (Weinhold) is helpful in most calculations (when Ruppeiner coordinates are complicated).

Some of our results have been supported by physical results in the literature in that *ultraspinning* instability of Myers-Perry Kerr black holes obtained from the curvature singularity of the Ruppeiner metric matches the results found earlier by Emparan and Myers.

State space diagrams can be very useful when studying extremal limits of BH solutions.

**Some issues**

1. Profusion of black hole solutions lead to incongruities of results (based on different versions of geometry)

2. BH phase transition. Who is least wrong?

3. (Quantum) statistical version of BH information geometry?