Determination of heat transfer coefficients

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Heat Transfer Law

\[ k \frac{\partial T}{\partial n} = \sigma (T_{\text{ambient}} - T)^\beta + B, \quad \text{on the boundary}\]

where

\( T = \) temperature
\( T_{\text{ambient}} = \) ambient temperature
\( k = \) thermal conductivity
\( n = \) outward unit normal to the boundary
\( B = \) additional heat flux
\( \sigma = \) heat transfer coefficient (may be space-, time-, or temperature-dependent)
\( \beta = 1 \) for convection; \( \beta = 4 \) for radiation.
Outline

Mathematical formulation
- Space-dependent heat transfer coefficient
- Time-dependent heat transfer coefficient
- Temperature-dependent heat transfer coefficient

Boundary element method (BEM)

Numerical Results and Discussion

Conclusions
1. Space-dependent Heat Transfer Coefficient
Consider the inverse problem which requires finding the temperature $T \in C^{2,1}(Q)$ and the space-dependent heat transfer coefficient $\sigma \in C(\partial \Omega), \sigma \geq 0$, satisfying the heat equation

$$\frac{\partial T}{\partial t}(x,t) = \nabla^2 T(x,t), \quad (x,t) = \Omega \times (0,t_f] =: Q,$$

subject to the initial condition

$$T(x,0) = T_0(x), \quad x \in \Omega,$$

the Robin boundary condition

$$\frac{\partial T}{\partial n}(x,t) + \sigma(x)T(x,t) = B(x,t), \quad (x,t) \in \partial\Omega \times (0,t_f),$$

and the instant temperature observation at the fixed time $t^0 \in (0,t_f)$:

$$T(x,t^0) = \chi(x), \quad x \in \partial\Omega$$

or, the additional integral time-average temperature observation

$$\int_0^{t_f} \omega(t)T(x,t)dt = \chi(x), \quad x \in \partial\Omega,$$

where $\omega \in L_1(0,t_f)$ is given.
Boundary Element Method (BEM)
Using the BEM we reduce the inverse problem to nonlinear boundary integral equations for the boundary temperature and the heat transfer coefficient:

\[
\frac{1}{2} T(x, t) = \int_{\Omega} G(x, t; y, 0) T_0(y) d\Omega(y) \\
+ \int_0^t \int_{\partial\Omega} B(\xi, \tau) G(x, t; \xi, \tau) dS(\xi) d\tau \\
- \int_0^t \int_{\partial\Omega} T(\xi, \tau) \left[ \frac{\partial G}{\partial n(\xi)}(x, t; \xi, \tau) + \sigma(\xi) G(x, t; \xi, \tau) \right] dS(\xi) d\tau,
\]

\[(x, t) \in \partial\Omega \times (0, t_f),\]

where

\[
G(x, t; \xi, \tau) = \frac{H(t - \tau)}{[4\pi(t - \tau)]^{n/2}} \exp \left( -\frac{\|x - \xi\|^2}{4(t - \tau)} \right)
\]

is the fundamental solution of the heat equation and \(H\) is the Heaviside function.
Numerical Example

Find the temperature \( T(x, t) \) \( (= x^2 + 2t) \) and the space-dependent heat transfer coefficient(s) \( 0 \leq \sigma_0 \ (= 1), \ 0 \leq \sigma_1 \ (= 1) \) solving the problem

\[
\frac{\partial T}{\partial t}(x, t) = \frac{\partial^2 T}{\partial x^2}(x, t), \quad (x, t) = (0, 1) \times (0, t_f = 1],
\]

\[
T(x, 0) = x^2, \quad x \in [0, 1],
\]

\[
- \frac{\partial T}{\partial x}(0, t) + \sigma_0 T(0, t) = 2t, \quad t \in (0, 1),
\]

\[
\frac{\partial T}{\partial x}(1, t) + \sigma_1 T(1, t) = 2t + 3, \quad t \in (0, 1),
\]

and the additional 1\% noisy measurement conditions

\[
T(0, t^0) = 2t^0 \times 1.01, \quad T(1, t^0) = (1 + 2t^0) \times 1.01,
\]

or

\[
\int_0^{t_f} T(0, t)dt = 1 \times 1.01, \quad \int_0^{t_f} T(1, t)dt = 2 \times 1.01.
\]

Using the BEM with \( N = N_0 = 40 \) elements, in the latter case we have obtained: \( \sigma_0 = 0.9875 \) and \( \sigma_1 = 0.9777 \). In the former case see the figure on the next slide.
Figure 6. The constants $\sigma_0 (\triangle)$ and $\sigma_1 (\square)$ for Problem I, as a function of $i_0 = 1, \ldots, N = 40$, when $(N_0, N) = (40, 40)$ (1% noise).
2. Time-dependent Heat Transfer Coefficient

Consider the inverse problem which requires finding the temperature $T \in C^{2,1}(Q)$ and the time-dependent heat transfer coefficient $\sigma \in C([0, t_f])$ satisfying the heat equation

$$\frac{\partial T}{\partial t}(x, t) = \nabla^2 T(x, t), \quad (x, t) = \Omega \times (0, t_f) =: Q,$$

subject to the initial condition

$$T(x, 0) = T_0(x), \quad x \in \Omega,$$

the Robin boundary condition

$$\frac{\partial T}{\partial n}(x, t) + \sigma(t)T(x, t) = B(x, t), \quad (x, t) \in \partial\Omega \times (0, t_f),$$

and the temperature observation at the fixed point $x_0 \in \partial\Omega$:

$$T(x_0, t) = \chi(t), \quad t \in [0, t_f]$$

or, the additional boundary integral temperature observation

$$\int_{\partial\Omega} \nu(x)T(x, t)dS(x) = \chi(t), \quad t \in [0, t_f],$$

where $\nu \in L_1(\partial\Omega)$ is given.
Numerical Example
Find the temperature $T(x, t) = x^2 + 2t + 1$ and the time-dependent heat transfer coefficient $\sigma(t) = t$, solving the problem

$$\frac{\partial T}{\partial t}(x, t) = \frac{\partial^2 T}{\partial x^2}(x, t), \quad (x, t) = (0, 1) \times (0, t_f = 1],$$

$$T(x, 0) = x^2, \quad x \in [0, 1],$$

$$- \frac{\partial T}{\partial x}(0, t) + \sigma(t)T(0, t) = 2t^2 + t, \quad t \in (0, 1),$$

$$\frac{\partial T}{\partial x}(1, t) + \sigma(t)T(1, t) = 2(t^2 + t + 1), \quad t \in (0, 1),$$

and the additional $\rho\%$ noisy measurement condition

$$T(0, t) = 2t + 1 + \epsilon, \quad t \in (0, 1),$$

where $\rho$ denotes the percentage of noise and $\epsilon$ are random variables taken from a Gaussian normal distribution with zero mean and standard deviation $3\rho\%$.

Using the BEM with $N = N_0 = 40$ elements and various amounts of noise $\rho\% \in \{1, 3, 5\}\%$ we obtain the figure on the next slide.
Figure 11. The analytical and numerical heat transfer coefficients $\sigma(t)$ for Problem II, as functions of time $t$, for various amounts of noise.
2’. Time-dependent Heat Transfer Coefficient
Consider the inverse problem which requires finding the pair solution
\((T(x, t), \sigma(t)) = (\text{temperature, heat transfer coefficient})\) satisfying the heat equation
\[
\frac{\partial T}{\partial t}(x, t) = \nabla^2 T(x, t), \quad (x, t) = \Omega \times (0, t_f) =: Q,
\]
subject to the initial condition
\[
T(x, 0) = T_0(x), \quad x \in \Omega,
\]
the Robin boundary conditions
\[
\frac{\partial T}{\partial n}(x, t) + \sigma(t)g(T(x, t)) = B(x, t), \quad (x, t) \in \partial \Omega \times (0, t_f),
\]
and the additional boundary integral (non-local) observation
\[
\int_{\partial \Omega} \Phi(T(x, t))dS(x) = E(t), \quad t \in [0, t_f],
\]
where \(\Phi(T) = \int^T g(s)ds\) denotes a primitive (anti-derivative) of \(g\).
Remarks:

- Of physical interest is the linear convection case $g(T) = T$, and the nonlinear radiative case $g(T) = T^3|T|$.

- Multiplying with $T$ the heat equation and integrating over $\Omega$ results in

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} T^2(x,t) d\Omega \right) + \int_{\Omega} |\nabla T|^2 d\Omega = \int_{\partial \Omega} T h dS - \sigma(t) \int_{\partial \Omega} g(T)T dS
\]

and one could recognise the last term as an ‘energy’ term

\[(\alpha + 1)\sigma(t)E(t)\]

for the nonlinearity $g(T) = T^\alpha$. 
Let us now consider the weak solutions $T$ and $\sigma$ of the inverse problem defined in the following spaces of functions:

$$T \in C([0, t_f], L^2(\Omega)) \cap L^2((0, t_f), H^1(\Omega))$$
with $\partial_t T \in L^2((0, t_f), L^2(\Omega))$.

$$\sigma \geq 0 \text{ and } \sigma \in C^1[0, t_f] \text{ with } \sigma'/\sigma \text{ bounded.}$$

We also require that the input data be such that:

$$T_0 \in H^2(\Omega), \quad B, B_t \in L^2((0, t_f), L^2(\partial \Omega)),$$
$$g' \geq 0, \quad g(0) = 0, \quad |g(s)| \leq C(|s|^{\alpha} + 1)$$

for some non-negative constants $C_0$, $C$ and $\alpha$. 
Definition. For a given $\sigma \in L_2(0, t_f)$, $\sigma \geq 0$, a function $T_\sigma \in L_2((0, t_f), H^1(\Omega))$ with $\partial_t T \in L_2((0, t_f), L_2(\Omega))$ is called a weak solution to the direct problem if $T_\sigma(x, 0) = T_0(x)$ and

$$(\partial_t T_\sigma, \phi) + (\partial_x T_\sigma, \partial_x \phi) + \sigma(g(T_\sigma), \phi)_{\partial \Omega} = (B, \phi)_{\partial \Omega},$$

$$\forall \phi \in H^1(\Omega), \text{ a.e. in } (0, t_f).$$

Theorem. (unique solvability of the direct problem)
There exists a unique weak solution to the direct problem.
Existence and Uniqueness Theorem. (Slodicka and Lesnic (2010))
Assume that a compatibility condition at $t = 0$ holds and that
$E'(t) \geq \delta_0 > 0$, $|E''(t)| \leq C_0$, $\forall t \in [0, t_f]$ and that

$$0 < E(t) \leq \int_{\partial \Omega} \Phi(T^0(x, t))dS(x), \quad \forall t \in [0, t_f],$$

where $T^0$ is the unique weak solution of the direct problem with $\sigma = 0$. Then there exists a unique solution to the inverse problem.

The continuous dependence of the solution on the input energy data $E(t)$ can (probably) be established under the additional assumption that $\sigma$ is bounded. This is an usual additional source condition which when imposed onto the solution of some ill-posed problems restore its stability with respect to noise added into the input data.
We employ the BEM

\[
\frac{1}{2}T(x, t) = \int_{\Omega} G(x, t; y, 0)T_0(y) d\Omega(y)
\]

\[
+ \int_0^t \int_{\partial \Omega} \left[ B(\xi, \tau) - g(T(\xi, \tau))\sigma(\tau) \right] G(x, t; \xi, \tau) dS(\xi) d\tau
\]

\[
- \int_0^t \int_{\partial \Omega} T(\xi, \tau) \frac{\partial G}{\partial n(\xi)} (x, t; \xi, \tau) dS(\xi) d\tau,
\]

\forall (x, t) \partial \Omega \times (0, t_f],

and

\[
\int_{\partial \Omega} \Phi(T(\xi, t)) dS(\xi) = E(t), \quad \forall t \in (0, t_f].
\]
Figure 2. The analytical and numerical boundary temperatures (a) $T(0, t)$ and (b) $T(1, t)$, the heat flow rate $q(0, t)$ and $q(1, t)$, the stress $\sigma(t)$, and (f) $E(t)$. Each plot has four curves for different values of $\rho$: $\rho = 0.10$, $\rho = 0.05$, $\rho = 0.00$, and the analytical solution.
3. Temperature-dependent Heat Transfer Coefficient

Consider the inverse problem of finding the temperature $T \in C^{3,3/2}(\overline{Q})$ and the space-dependent heat transfer coefficient $\sigma \in C^1([\theta_1, \theta_2])$, where $\theta_1 = \min_Q u(x, t)$ and $\theta_2 = \max_Q u(x, t)$ are assumed known a priori and satisfy $\theta_1 \theta_2 > 0$. We also assume

$$\overline{\chi}(0) \leq u(x, t) \leq \overline{\chi}(t), \quad (x, t) \in \partial \Omega \times [0, t_f].$$

In addition, the pair solution $(T, \sigma(T))$ satisfies the heat equation

$$\frac{\partial T}{\partial t}(x, t) = \nabla^2 T(x, t), \quad (x, t) = \Omega \times (0, t_f) =: Q,$$

subject to the initial condition

$$T(x, 0) = T_0(x), \quad x \in \Omega,$$

the Robin boundary condition

$$\frac{\partial T}{\partial n}(x, t) + \sigma(T(x, t))T(x, t) = B(x, t), \quad (x, t) \in \partial \Omega \times (0, t_f),$$

and the temperature observation at the fixed point $x_0 \in \partial \Omega$:

$$T(x_0, t) = \overline{\chi}(t), \quad t \in [0, t_f].$$
**Uniqueness Theorem.** (Rundell and Yin (1990))

If $B \in C^{2,2}(\partial\Omega \times [0, t_f])$, and $\chi \in C^{2}([0, t_f])$ is strictly increasing, then the solution is unique.

Further, in the one-dimensional case we seek $T \in C^{2,1}(Q)$ and

$$\sigma \in \Sigma_{adm} := \{\sigma \in C^{0+1}([\theta_1, \theta_2]) | 0 < m_1 \leq \sigma(T) \leq M_1 < \infty\},$$

where $\theta_1 = min\{0, \inf_{x \in (0,1)} T_0(x)\}$ and $\theta_2 = max\{0, \max_{x \in (0,1)} T_0(x)\}$.

**Existence and Uniqueness Theorem.** (Pilant and Rundell (1989))

In the one-dimensional case, if $T_0 \in C^{2+1/2}([0, 1])$, $B = 0$, and $\chi \in C^{1+1/2}([0, t_f])$ is strictly monotone and $\chi(0) = T_0(0) = T_0(1)$, then the inverse problem has a unique solution.
Boundary Element Method (BEM)
Using the BEM we reduce the inverse problem to nonlinear boundary integral equations for the boundary temperature and the heat transfer coefficient:

\[
\frac{1}{2} T(x, t) = \int_{\Omega} G(x, t; y, 0) T_0(y) d\Omega(y) \\
+ \int_{0}^{t} \int_{\partial\Omega} B(\xi, \tau) G(x, t; \xi, \tau) dS(\xi) d\tau \\
- \int_{0}^{t} \int_{\partial\Omega} T(\xi, \tau) \left[ \frac{\partial G}{\partial n(\xi)}(x, t; \xi, \tau) + \sigma(T(\xi, \tau)) G(x, t; \xi, \tau) \right] dS(\xi) d\tau,
\]

\((x, t) \in \partial\Omega \times (0, t_f)\).
In one-dimension, with the temperature measurement taken at the boundary point \( x_0 = 0 \) we obtain a coupled system of two nonlinear boundary integral equations in two unknowns, namely \( T(1, t) \) and \( \sigma(T(1, t)) \):

\[
\frac{1}{2} \bar{\chi}(t) = \int_0^1 G(0, t; y, 0) T_0(y) dy \\
+ \int_0^t \bar{\chi}(t) \left[ G(0, t; 0, \tau) \sigma(\bar{\chi}(t)) + \frac{\partial G}{\partial \xi}(0, t; 0, \tau) \right] d\tau \\
+ \int_0^t T(1, t) \left[ G(0, t; 1, \tau) \sigma(\bar{\chi}(t)) - \frac{\partial G}{\partial \xi}(0, t; 1, \tau) \right] d\tau, \quad t \in (0, t_f),
\]

\[
\frac{1}{2} T(1, t) = \int_0^1 G(1, t; y, 0) T_0(y) dy \\
+ \int_0^t \bar{\chi}(t) \left[ G(1, t; 0, \tau) \sigma(\bar{\chi}(t)) + \frac{\partial G}{\partial \xi}(1, t; 0, \tau) \right] d\tau \\
+ \int_0^t T(1, t) \left[ G(1, t; 1, \tau) \sigma(T(1, t)) - \frac{\partial G}{\partial \xi}(1, t; 1, \tau) \right] d\tau, \quad t \in (0, t_f).
\]
Using a constant BEM approximation with $N$ boundary elements and $N_0$ cells, we obtain a system of $2N$ nonlinear equations

$$A_\sigma(T_1) = b,$$

where $T_1$ contains $T(1, t)$, $b$ contains $T_0$ and $B$, and $A_\sigma$ is a nonlinear operator depending on $\sigma$. Assuming that $\overline{\chi}$ is strictly increasing, let

$$q_k := \overline{\chi}(0) + k(\overline{\chi}(t_f) - \overline{\chi}(0))/K, \quad k = 0, K$$

denote a uniform discretisation of the interval $[\overline{\chi}(0), \overline{\chi}(t_f)]$ into $K$ equal sub-intervals. Then we seek a piecewise constant function $\sigma(T)T =: f : [q_0, q_K] \rightarrow \mathbb{R}$ defined by

$$f(T) = \begin{cases} 
  a_1, & T \in [q_0, q_1) \\
  a_2, & T \in [q_1, q_2) \\
  \vdots & \vdots \\
  a_K, & T \in [q_{K-1}, q_K)
\end{cases}$$

where the unknown coefficients $\underline{a} = (a_k)_{k=1,K}$ are yet to be determined.
Assuming $T(1, t; f) \in [\overline{x}(0), \overline{x}(t)]$, $\forall t \in [0, t_f]$, we have

$$f(\overline{x}(\tilde{t}_l)) = a_{\phi(l)}, \quad f(T(1, \tilde{t}_l)) = a_{\psi(l)}, \quad l = 1, l,$$

where $\tilde{t}_l$ are the boundary element nodes, and for each $l \in \{1, ..., N\}$, $\phi(l)$ is the unique number in the set $\{1, ..., K\}$ such that $\overline{x}(\tilde{t}_l) \in [q_{\phi(l)} - 1, q_{\phi(l)})$, and $\psi(l)$ is the unique number in the set $\{1, ..., K\}$ such that $T(1, \tilde{t}_l) \in [q_{\psi(l)} - 1, q_{\psi(l)})$.

We then minimize (using the NAG routine E04FCF) the nonlinear Tikhonov functional

$$S : \mathbb{R}^K \times \mathbb{R}^N \to \mathbb{R}_+, \quad S(a, T_1) := \|A_\sigma(T_1) - b\|^2 + \kappa\|a\|^2,$$

where $\kappa > 0$ is a regularization parameter to be prescribed.
Numerical Examples
The BEM is applied with \((N, N_0) = (40, 40)\) to generate the forward operator \(A_\sigma(T_1)\). The piecewise constant parametrisation of \(f(T) = \sigma(T)T\) is sought with \(K = 10\).

The analytical temperature to be retrieved

\[
T(x, t) = x^2 - x + 1 + 2t, \quad (x, t) \in [0, 1] \times [0, 1],
\]

generates the initial temperature

\[
T(x, 0) = T_0(x) = x^2 - x + 1, \quad x \in [0, 1],
\]

and the boundary temperature measurement

\[
T(0, t) = \overline{\chi}(t) = 1 + 2t, \quad t \in [0, 1].
\]

Remark that \(\overline{\chi}\) is strictly increasing and that

\[
T(1, t) = 1 + 2t \in [\overline{\chi}(0) = 1, \overline{\chi}(t) = 1 + 2t], \quad \forall t \in [0, 1],
\]

such that the unique solvability of the inverse problem is ensured.

Numerical results are presented next for \(f(T) \in \{1, T^4\}\) which corresponds to a heat transfer coefficient \(\sigma(T) \in \{T^{-1}, T^3\}\).
Example 1. $f(T) = 1$. Initial guess $(a^0, T^0_1) = (3, 3)$.

Figure 1: (a) The analytical boundary temperature $T(0, t)$, (b) the numerical boundary temperature $T(1, t)$, as functions of time $t$, and (c) the numerical vector $a = (a_k)_{k=T_0}$, when the amount of noise in (4.4) is: (a)
Example 2. \( f(T) = T^4 \). Initial guess \((a^0, T_1^0) = (50, 3)\) and the Neumann conditions (1) and (2) modified as

\[
\frac{\partial T}{\partial n}(x, t) = f(T(x, t)) + 1 - (1 + 2t)^4, \quad (x, t) \in \{0, 1\} \times (0, 1].
\]

Figure 2: The analytical and numerical approximations of (a) the boundary temperature \( T(1, t) \) and (b) the function \( f(T) \), when \( \rho \in \{0, 1, 3, 5\} \% \) and \( \kappa = 0 \).

Figure 3: The analytical and numerical approximations of (a) the boundary temperature \( T(1, t) \) and (b) the function \( f(T) \), when \( \rho = 5\% \), \( \kappa = 0 \) and \( 10^{-3} \).
4. Conclusions

- Reconstruction of heat transfer coefficient which may be space-, time-, or temperature-dependent has been addressed.

- The existence and uniqueness of solution has been discussed in both strong and weak senses. Furthermore, a numerical method based on the boundary element method (BEM) (combined with the Tikhonov regularization method where necessary) has been devised in order to obtain stable and accurate numerical solutions.

- Future work will investigate iterative regularizations and higher-dimensional numerical reconstructions.