Acceleration of the modified alternating algorithm by the conjugate gradient method for the Cauchy problem for the Helmholtz equation

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- Cauchy problem for the Helmholtz equation
- Alternating iterative algorithm
- Modified alternating algorithm
- Conjugate gradient method
Formulation of the Cauchy Problem for the Helmholtz equation

- Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with a Lipschitz boundary \( \Gamma \).
- The boundary \( \Gamma \) is divided into two parts \( \Gamma_0 \) and \( \Gamma_1 \).

Consider the Cauchy problem for the Helmholtz equation:

\[
\begin{aligned}
\Delta u + k^2 u &= 0 \quad \text{in} \quad \Omega, \\
\quad u &= f \quad \text{on} \quad \Gamma_0, \\
\partial_\nu u &= g \quad \text{on} \quad \Gamma_0,
\end{aligned}
\]

where \( k \) is the wave number.

- The problem is ill-posed.
- Applications: characterization of sound sources (Langrenne and Garcia: 2011), . . .
Alternating algorithm

Following


the alternating algorithm may be described in the following way:

\[
\begin{align*}
\Delta u + k^2 u &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \Gamma_0, \\
\partial_\nu u &= \eta \quad \text{on } \Gamma_1,
\end{align*}
\]

(1)

\[
\begin{align*}
\Delta u + k^2 u &= 0 \quad \text{in } \Omega, \\
\partial_\nu u &= g \quad \text{on } \Gamma_0, \\
u &= \phi \quad \text{on } \Gamma_1,
\end{align*}
\]

(2)

The first approximation \(u_0\) to the solution \(u\) is obtained by solving (1), where \(\eta\) is an arbitrary initial approximation of the normal derivative on \(\Gamma_1\).

Having constructed \(u_{2n}\), we find \(u_{2n+1}\) by solving (2) with \(\phi = u_{2n}\) on \(\Gamma_1\).

We then obtain \(u_{2n+2}\) by solving the problem (1) with \(\eta = \partial_\nu u_{2n+1}\) on \(\Gamma_1\).
Previous works


Nonconvergence of the original algorithm for the Cauchy problem for the Helmholtz equation

Consider the Cauchy problem for the Helmholtz equation in a rectangle \([0, a] \times [0, b]\):

\[
\begin{aligned}
\Delta u(x, y) + k^2 u(x, y) &= 0, & 0 < x < a, & 0 < y < b, \\
u(x, 0) &= f(x), & 0 \leq x \leq a, \\
u_y(x, 0) &= g(x), & 0 \leq x \leq a, \\
u(0, y) &= u(a, y) = 0, & 0 \leq y \leq b.
\end{aligned}
\]

This problem is ill–posed.

The algorithm diverges for

\[
k^2 \geq \pi^2 \left( a^{-2} + (4b)^{-2} \right)
\]
Choice of the interior boundary


- Introduce open subsets $\omega_i$, $i = 1, \ldots, n$ inside $\Omega$ with boundaries $\gamma_i$, $i = 1, \ldots, n$.
- We assume that every $\omega_i$ is a Lipschitz domain.

\[ \Omega_1 = \bigcup_{i=1}^{n} \omega_i \]  
with Lipschitz boundary $\gamma = \bigcup_{i=1}^{n} \gamma_i$ and  
\[ \Omega_2 = \Omega \setminus (\Omega_1 \cup \gamma). \]
Choice of the interior boundary $\gamma$ and the constant $\mu$

**Assumption:** For all non-zero $u$,

$$
\int_{\Omega} (|\nabla u|^2 - k^2 u^2) \, dx + \mu \int_{\gamma} u^2 \, dS > 0,
$$

for $u \in H^1(\Omega)$ such that $u \neq 0$. 

**Theorem**

Let

\[ \Lambda_\mu = \min_{u \in H^1(\Omega), \|u\|_2 = 1} \int_\Omega |\nabla u|^2 \, dx + \mu \int_\gamma u^2 \, dS, \]

and

\[ \Lambda = \min_{u \in H^1(\Omega), u|_\gamma = 0, \|u\|_2 = 1} \int_\Omega |\nabla u|^2 \, dx. \]

Then there exists a positive constant \( C \) such that

\[ \Lambda - \Lambda_\mu \leq \frac{C(\Lambda)^{3/2}}{\mu^{1/2}}. \]
Corollary

If $\Lambda$ is positive, then
\[
\int_{\Omega} \left( |\nabla u|^2 - k^2 u^2 \right) \, dx + \mu \int_{\gamma} u^2 \, dS > 0, \quad \text{for all } u, \quad u \neq 0 \quad \text{on } \gamma.
\]

for sufficiently large $\mu$. 

Modified alternating iterative algorithm for the Cauchy problem for the Helmholtz equation

The modified algorithm will consist of solving the following well–posed problems alternatively:

\[
\begin{aligned}
\Delta u + k^2 u &= 0 \quad \text{in } \Omega \setminus \gamma, \\
u &= f \quad \text{on } \Gamma_0, \\
\partial_\nu u &= \eta \quad \text{on } \Gamma_1, \\
[u] + \mu u &= \xi \quad \text{on } \gamma, \\
[u] &= 0 \quad \text{on } \gamma,
\end{aligned}
\]

\[
\begin{aligned}
\Delta v + k^2 v &= 0 \quad \text{in } \Omega \setminus \gamma, \\
\partial_\nu v &= g \quad \text{on } \Gamma_0, \\
v &= \phi \quad \text{on } \Gamma_1, \\
v &= \varphi \quad \text{on } \gamma.
\end{aligned}
\]

The first approximation \(u_0\) to the solution \(u\) is obtained by solving (3), where \(\eta\) is an arbitrary initial approximation of the normal derivative on \(\Gamma_1\) and \(\xi\) is an arbitrary approximation of \([\partial_\nu u] + \mu u\) on \(\gamma\).

Having constructed \(u_{2n}\), we find \(u_{2n+1}\) by solving (4) with \(\phi = u_{2n}\) on \(\Gamma_1\) and \(\varphi = u_{2n}\) on \(\gamma\).

We then obtain \(u_{2n+2}\) by solving the problem (3) with \(\eta = \partial_\nu u_{2n+1}\) on \(\Gamma_1\) and \(\xi = [\partial_\nu u_{2n+1}] + \mu u_{2n+1}\) on \(\gamma\).
Convergence of the modified alternating iterative algorithm

**Theorem**

Let \( f \in H^{1/2}(\Gamma_0) \) and \( g \in H^{1/2}(\Gamma_0)^* \), and let \( u \in H^1(\Omega) \) be the solution to the Cauchy problem for the Helmholtz equation given above. Then, for every \( \eta \in H^{1/2}(\Gamma_1)^* \) and every \( \xi \in H^{1/2}(\gamma)^* \), the sequence \((u_n)_{n=0}^{\infty}\) obtained from the modified alternating algorithm converges to \( u \) in \( H^1(\Omega) \).
Given $\eta \in H^{1/2}(\Gamma_1)^*$ and $\xi \in H^{1/2}(\gamma)^*$, let us define

$$B(\eta, \xi) = (\partial_\nu v |_{\Gamma_1}, [\partial_\nu v] + \mu v |_{\gamma}).$$

We find that

$$(\eta_{k+1}, \xi_{k+1}) = B(\eta_k, \xi_k).$$
Consider the following problem

\[
\begin{cases}
\Delta u + k^2 u = 0 & \text{in } \Omega \backslash \gamma, \\
u = 0 & \text{on } \Gamma_0, \\
\partial_\nu u = \eta & \text{on } \Gamma_1, \\
[\partial_\nu u] + \mu u = \xi & \text{on } \gamma, \\
[u] = 0 & \text{on } \gamma,
\end{cases}
\]

Introduce a linear operator \( N : H^{1/2}(\Gamma_1)^* \times H^{1/2}(\gamma)^* \to H^{1/2}(\Gamma_0)^* \) by

\[
N(\eta, \xi) = \partial_\nu u \big|_{\Gamma_0},
\]

where \( \eta \in H^{1/2}(\Gamma_1)^* \), \( \xi \in H^{1/2}(\gamma)^* \).

If \( u \in H^1(\Omega) \) solves the Cauchy problem for the Helmholtz equation with \( f = 0 \) on \( \Gamma_0 \), the problem can then be formulated as

\[
N(\eta, \xi) = g.
\]
Adjoint operator $N^*$

**Lemma**

Let $\zeta \in H^{1/2}(\Gamma_0)^*$, and let $v$ solves the

\[
\begin{cases}
\Delta w + k^2 w = 0 & \text{in } \Omega \setminus \gamma, \\
\partial_\nu w = \zeta & \text{on } \Gamma_0, \\
w = 0 & \text{on } \Gamma_1, \\
w = 0 & \text{on } \gamma.
\end{cases}
\]

Then $N^*(\zeta) = (\partial_\nu w|_{\Gamma_1}, [\partial_\nu w] + \mu w|_{\gamma})$. 
Consider the following functional

\[ J(\eta, \xi) = \|g - N(\eta, \xi)\|_{H^{1/2}(\Gamma_0)} \]

Let us define

\[ L_N(\eta, \xi) = (\eta, \xi) + \alpha N^*(g - N(\eta, \xi)), \]

where \( \alpha \) is a fixed constant chosen so that \( 0 < \alpha < \|N\|^{-2} \).

The Landweber method produces iterates

\[ (\eta_{k+1}, \xi_{k+1}) = L_N(\eta_k, \xi_k). \]
Theorem

For any $\eta \in H^{1/2}(\Gamma_1)^*$ and $\xi \in H^{1/2}(\gamma)$, the iterates produced by the Landweber method and the modified alternating algorithm are identical, i.e.,

$$L_N(\eta, \xi) = B(\eta, \xi).$$

(5)
The conjugate gradient method for the problem is as follows

1. Choose initial $\eta_0 \in H^{1/2}(\Gamma_1)^*$ and $\xi_0 \in H^{1/2}(\gamma)^*$. Denote $\chi_0 = (\eta_0, \xi_0)$ and $(H^{1/2})^* = H^{1/2}(\Gamma_1)^* \times H^{1/2}(\gamma)^*$.
2. $d_0 = g - N(\chi_0)$;
3. $p_1 = s_0 = N^*(d_0)$;
4. for $k = 1, 2, \ldots$, unless $s_{k-1} = 0$, compute:
   5. $q_k = N(\chi_k)$;
   6. $\alpha_k = \|s_{k-1}\|_{(H^{1/2})^*}/\|q_k\|_{H^{1/2}(\Gamma_0)^*}$;
   7. $\chi_k = \chi_{k-1} + \alpha_k p_k$;
   8. $d_k = d_{k-1} - \alpha_k q_k$;
   9. $s_k = N^*(d_k)$;
   10. $\alpha_k = \|s_k\|_{(H^{1/2})^*}/\|s_{k-1}\|_{(H^{1/2})^*}$;
   11. $p_{k+1} = s_k + \beta_k p_k$. 


Numerical experiments

- The domain is the rectangle $\Omega = (0, 1) \times (0, L)$.
- We put $\Gamma_0 = (0, 1) \times \{0\}$ and $\Gamma_1 = (0, 1) \times \{L\}$.
- We choose $L = 0.2$, the computational grid $N = 401$, and $M = 81$ and the following exact data:

$$u(x, 0) = \left( 3 \sin \pi x + \frac{\sin 3\pi x}{19} + 9 \exp(-30(x - L)^2) \right) x^2 (1 - x)^2,$$

and

$$u(x, L) = 2 \left( 8 \sin \pi x + \frac{\sin 3\pi x}{17} + 20 \exp(-50(x - L)^2) \right) x^2 (1 - x)^2.$$
Numerical experiments

Figure 1: Modified algorithm (left) after 1500 iterations and the conjugate gradient method (right) after 20 iterations.
THANK YOU FOR YOUR ATTENTION.