Geometry of optimal decomposition for the $L$-functional and duality in convex analysis

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Conference on "Inverse Problems and Applications"

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April 06, 2013
ROF model

In 1992 Rudin, Osher and Fatemi suggested a denoising model which has made great success. Let $D = [a, b] \times [c, d]$ be a rectangular domain in $\mathbb{R}^2$. Suppose that initial image $f_* \in BV$ and we observe

$$f_{ob} = f_* + \eta,$$

where $\eta \in L^2(D)$ corresponds to noise. In order to reconstruct approximately initial image $f_*$, ROF suggested to consider

$$L_{2,1} \left( t, f_{ob}, L^2(D), BV(D) \right) = \inf_{g \in BV} \left( \frac{1}{2} \left\| f_{ob} - g \right\|_{L^2}^2 + t \left\| g \right\|_{BV} \right),$$

and to take as approximation to $f_*$ the function $f_t$ which minimizes this functional, i.e.,
**ROF model**

\[ L_{2,1} (t, f_{ob}, L^2(D), BV(D)) = \frac{1}{2} \| f_{ob} - f_t \|_{L^2}^2 + t \| f_t \|_{BV}, \]

where (for a function \( f \) of class \( C^1 \))

\[ \| f \|_{BV} = \iint_D \left( \left| \frac{\partial f}{\partial x} (x, y) \right| + \left| \frac{\partial f}{\partial y} (x, y) \right| \right) \, dx dy. \]

**Optimal decomposition**

The expression

\[ f_{ob} = (f_{ob} - f_t) + f_t, \]

is called *optimal decomposition* of \( L_{2,1} (t, f_{ob}, L^2(D), BV(D)) \) corresponding to \( f_{ob} \).
In 2002, Yves Meyer obtained a mathematical characterization of this optimal decomposition for this couple \((L^2(D), BV(D))\) by using duality. [Yves Meyer, *Oscillating Patterns in Image Processing and Nonlinear Evolution Equations*, 2002]
Let \((X_0, X_1)\) be a compatible Banach couple. i.e., \(X_0\) and \(X_1\) are Banach spaces such that \(X_0\) and \(X_1\) are linearly and continuously embedded in some Banach space \(X\). Let \(x \in X_0 + X_1\), let \(1 < p < +\infty\) and \(t > 0\). We consider the \(L\)-functional

\[
L_{p,1} (t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left( \frac{1}{p} \|x_0\|_{X_0}^p + t \|x_1\|_{X_1} \right),
\]

We give a characterization of \textit{optimal decomposition} for the \(L\)-functional. i.e., \(x = x_{0,\text{opt}} + x_{1,\text{opt}}\) such that

\[
L_{p,1} (t, x; X_0, X_1) = \frac{1}{p} \|x_{0,\text{opt}}\|_{X_0}^p + t \|x_{1,\text{opt}}\|_{X_1}
\]
The problem

\[ L_{p,1}(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left( \frac{1}{p} \|x_0\|_{X_0}^p + t \|x_1\|_{X_1} \right), \]

We give a characterization of optimal decomposition for the L-functional. i.e., \( x = x_{0,\text{opt}} + x_{1,\text{opt}} \) such that

\[ L_{p,1}(t, x; X_0, X_1) = \frac{1}{p} \|x_{0,\text{opt}}\|_{X_0}^p + t \|x_{1,\text{opt}}\|_{X_1} \]

For ROF model,

\[ p = 2, \ X_0 = L^2(D), \ X_1 = BV(D) \]
Dual characterization of optimal decomposition

Let \((X_0, X_1)\) be a regular couple \((X_0 \cap X_1 \text{ is dense in both } X_0 \text{ and } X_1)\). Then it is a known fact from interpolation theory that \((X_0^*, X_1^*)\) also form a Banach couple and \((X_0 \cap X_1)^* = X_0^* + X_1^*\).

The dual spaces are defined by the norm

\[
\|y\|_{X_j^*} = \sup \left\{ \langle y, x \rangle : x \in X_j, \|x\|_{X_j} \leq 1 \right\}, \quad j = 0, 1.
\]

The spaces \(X_0 + X_1\) and \(X_0 \cap X_1\) are Banach spaces with respect to the following norms

\[
\|x\|_{X_0 + X_1} = \inf_{x = x_0 + x_1} \left\{ \|x_0\|_{X_0} + \|x_1\|_{X_1} \right\},
\]

where the infimum extends over all representations \(x = x_0 + x_1\) of \(x\) with \(x_0\) in \(X_0\) and \(x_1\) in \(X_1\), and

\[
\|x\|_{X_0 \cap X_1} = \max \left\{ \|x\|_{X_0}, \|x\|_{X_1} \right\}.
\]
Theorem (Main Theorem)  

Let $1 < p < +\infty$. The decomposition $x = x_{0, opt} + x_{1, opt}$ is optimal for $L_{p,1}(t, x; X_0, X_1)$ if and only if $\exists y_* \in X_0^* \cap X_1^*$ such that $\|y_*\|_{X_1^*} \leq t$ and

$$\begin{align*}
\frac{1}{p} \|x_{0, opt}\|_{X_0} &= \langle y_*, x_{0, opt} \rangle - \frac{1}{p'} \|y_*\|_{X_0^*}^p;
\frac{1}{p'} \|x_{1, opt}\|_{X_1} &= \langle y_*, x_{1, opt} \rangle,
\end{align*}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. 

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**Dual characterization of optimal decomposition**

**Introduction**

Optimal decomposition for couple $(X_0, X_1)$

Optimal decomposition for couple $(\ell^p, X)$

**General case**

Dual characterization of optimal decomposition for

**Theorem (Main Theorem)**

Let $1 < p < +\infty$. The decomposition $x = x_{0, opt} + x_{1, opt}$ is optimal for $L_{p,1}(t, x; X_0, X_1)$ if and only if $\exists y_* \in X_0^* \cap X_1^*$ such that $\|y_*\|_{X_1^*} \leq t$ and

$$\begin{align*}
\frac{1}{p} \|x_{0, opt}\|_{X_0} &= \langle y_*, x_{0, opt} \rangle - \frac{1}{p'} \|y_*\|_{X_0^*}^p;
\frac{1}{p'} \|x_{1, opt}\|_{X_1} &= \langle y_*, x_{1, opt} \rangle,
\end{align*}$$

where $\frac{1}{p} + \frac{1}{p'} = 1.$
Consider a particular but important case of couple \((\ell^p, X)\) on \(\mathbb{R}^n\)

\[
L_{p,1} (t, x; \ell^p, X) = \inf_{x = x_0 + x_1} \left( \frac{1}{p} \| x_0 \|_{\ell^p}^p + t \| x_1 \|_X \right),
\]

where \(1 < p < +\infty\). Consider the following function

\[
F_0 (u) = \frac{1}{p} \| u \|_{\ell^p}^p, \quad \nabla F_0 (v) = \left\{ |v|^{p-1} \text{sgn} (v) \right\}
\]

Let us define the set \(\Omega\) by

\[
\Omega = \{ v \in \mathbb{R}^n : \nabla F_0 (v) \in tB_{X^*} \},
\]
Couple \((\ell^p, X)\)

\[ \Omega = \{ v \in \mathbb{R}^n : \nabla F_0(v) \in tB_{X^*} \} \]. There are two cases

**Case 1:** If \( x \in \Omega \)
then the optimal decomposition for \( L_{p,1}(t, x; \ell^p, X) \) is given by

\[ x_{0, \text{opt}} = x \text{ and } x_{1, \text{opt}} = 0 \]

**Case 2:** If \( x \notin \Omega \)

**Theorem**

*The decomposition \( x = x_{0, \text{opt}} + x_{1, \text{opt}} \) is optimal for \( L_{p,1}(t, x; \ell^p, X) \) if and only if*

(a) \( \| \nabla F_0(x_{0, \text{opt}}) \|_{X^*} = t \)

(b) \( \langle x_{1, \text{opt}}, \nabla F_0(x_{0, \text{opt}}) \rangle = t \| x_{1, \text{opt}} \|_X \).
Geometry of optimal decomposition for couple \((\ell^p, X)\)

\(x_{1,\text{opt}}\) is orthogonal to the supporting hyperplane to \(tB_{X^*}\) at \(y_*\)
Case $p = 2$: Couple $(\ell^2, X)$

$$F_0(u) = \frac{1}{2} \|u\|_{\ell^2}^2, \quad \nabla F_0(v) = v$$

The sets $\Omega$ and $tB_{X^*}$ coincide

$$\Omega = tB_{X^*} = \{ u \in \mathbb{R}^n : \|u\|_{X^*} \leq t \}$$

**Corollary (for $x \notin \Omega$)**

$$\|x_{0, opt}\|_{X^*} = t \text{ and } \langle x_{0, opt}, x - x_{0, opt} \rangle = t \|x - x_{0, opt}\|_{X}.$$
Case $p = 2$: Couple $(\ell^2, X)$

**Theorem**

Let $x_{0, opt}$ be an exact minimizer for $L_{2,1}(t, x; \ell^2, X)$. Then $x_{0, opt}$ is the nearest element of $tB_X^*$ to the point $x$ in the metric of $\ell^2$:

$$E(t, x; \ell^2, X^*) = \inf_{\|x_0\|_{X^*} \leq t} \|x - x_0\|_{\ell^2} = \|x - x_{0, opt}\|_{\ell^2}.$$
Geometry of optimal decomposition for couple \((\ell^2, X)\)
Illustration in the plane

Consider couple \((\ell^3, X)\) in the plane where the unit ball of \(X\) is the rotated ball of \(\ell^1\) by the rotation matrix

\[
R_\theta = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix},
\]

for \(\theta = 30^\circ\). We have that

\[
\|x\|_X = \|R_\theta^{-1}x\|_{\ell^1} = \left| \frac{\sqrt{3}}{2} x_1 - \frac{1}{2} x_2 \right| + \left| \frac{1}{2} x_1 + \frac{\sqrt{3}}{2} x_2 \right|.
\]

\[
\nabla F_0(u) = \left[ |u_1|^2 \text{sgn}(u_1), |u_2|^2 \text{sgn}(u_2) \right].
\]

The set \(\Omega\) can be written as

\[
\Omega = \left\{ v \in \mathbb{R}^2 : \left\| \begin{bmatrix} |v_1|^2 \text{sgn}(v_1), |v_2|^2 \text{sgn}(v_2) \end{bmatrix}^T \right\|_{X^*} \leq t \right\},
\]
Illustration in the plane

where the norm in $X^*$ is given by

$$\|y\|_{X^*} = \|R_\theta y\|_{\ell_\infty} = \max \left\{ \left| \frac{\sqrt{3}}{2} y_1 + \frac{1}{2} y_2 \right|, \left| -\frac{1}{2} y_1 + \frac{\sqrt{3}}{2} y_2 \right| \right\}.$$
Illustration in the plane

Geometry of Optimal Decomposition for the Couple \((\ell_p, X)\) for \(p = 3\), \(X = R_\theta (\ell_1)\) and \(\theta = 30^\circ\). The set \(\Omega\) could be of rather complicated structure.
Dual characterization of optimal decomposition

**Theorem (general case)**

Let \( x \in X_0 + X_1 \), \( 1 < p_0, p_1 < \infty \) and let \( t > 0 \) be a fixed parameter. The decomposition \( x = x_{0,\text{opt}} + x_{1,\text{opt}} \) is optimal for

\[
L_{p_0,p_1}(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left( \frac{1}{p_0} \| x_0 \|_{X_0}^{p_0} + \frac{t}{p_1} \| x_1 \|_{X_1}^{p_1} \right),
\]

if and only if \( \exists y_\ast \in X_0^* \cap X_1^* \) such that

\[
\begin{cases}
\frac{1}{p_0} \| x_{0,\text{opt}} \|_{X_0}^{p_0} = \langle y_\ast, x_{0,\text{opt}} \rangle - \frac{1}{p_0} \| y_\ast \|_{X_0^*}^{p_0'}; \\
\frac{t}{p_1} \| x_{1,\text{opt}} \|_{X_1}^{p_1} = \langle y_\ast, x_{1,\text{opt}} \rangle - \frac{t}{p_1} \| y_\ast \|_{X_1^*}^{p_1'}.
\end{cases}
\]
Thank you for your attention!