Hypothesis testing for a separable covariance structure with AR(1) under the two-level multivariate model

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LinStat 2014, Linköping, Sweden

Data

- q – number of characteristics

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- p – number of time points

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- n – number of individuals

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- n – number of individuals

- $\mathbf{X}_i,\ i = 1, \ldots, n$ – i.i.d. observation matrices

Observation matrices

$$\mathbf{X}_1 = \begin{pmatrix} x_{1,1,1} & x_{1,1,2} & \cdots & x_{1,1,p} \\ x_{1,2,1} & x_{1,2,2} & \cdots & x_{1,2,p} \\ \vdots & \vdots & \ldots & \vdots \\ x_{1,q,1} & x_{1,q,2} & \cdots & x_{1,q,p} \end{pmatrix}$$

$$\vdots$$

$$\mathbf{X}_n = \begin{pmatrix} x_{n,1,1} & x_{n,1,2} & \cdots & x_{n,1,p} \\ x_{n,2,1} & x_{n,2,2} & \cdots & x_{n,2,p} \\ \vdots & \vdots & \ldots & \vdots \\ x_{n,q,1} & x_{n,q,2} & \cdots & x_{n,q,p} \end{pmatrix}$$

Two-level multivariate model

Model

$$\mathbf{X}_i \sim N_{q,p}(\mathbf{M}, \Omega)$$

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- $\mathbf{X}_i$ - matrix of observations
- $\mathbf{M}$ - matrix of means
- Ω - variance-covariance matrix (p.d.)

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$$\mathrm{vec}\mathbf{X}_i \sim N_{pq}(\mathrm{vec}\mathbf{M}, \Omega)$$

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$$\text{vec}\mathbf{X}_i \sim N_{pq}(\text{vec}\mathbf{M}, \Omega)$$

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Vector of unknown parameters:

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \mu \\ \mathrm{vech}\Omega \end{pmatrix}$$

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Estimability of Ω: $n > pq$

Hypothesis

$$\underset{pq\times pq}{\Omega} = \underset{p\times p}{\Psi} \otimes \underset{q\times q}{\Sigma}$$

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Number of parameters: $\frac{p(p+1)}{2} + \frac{q(q+1)}{2} - 1$

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Dutilleul (1999), Roy & Khattree (2003), Lu & Zimmerman (2005), Roy (2007), Srivastava et al. (2008), Werner et al. (2008)

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$$\Psi = \begin{pmatrix} 1 & \rho & \rho^2 & \dots & \rho^{p-1} \\ \rho & 1 & \rho & \dots & \rho^{p-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{p-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{p-1} & \rho^{p-2} & \rho^{p-3} & \dots & 1 \end{pmatrix}$$

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Roy and Khatree, 2005, Roy and Leiva, 2008

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Degrees of freedom:

$$\nu = \frac{pq(pq+1)}{2} - \frac{q(q+1)}{2} - 1$$

Likelihood ratio test

- $\mathbf{X} = [\mathrm{vec}\mathbf{X}_1, \mathrm{vec}\mathbf{X}_2, \ldots, \mathrm{vec}\mathbf{X}_n] \in \mathbb{R}^{pq,n}$ – data matrix
- $\ln L(\mu, \Omega; \mathbf{X})$ – log-likelihood function (partially differentiable with respect to each coordinate of θ for every $\mathbf{X}$).

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Likelihood ratio (LR)

$$\Lambda = \frac{\max_{H_0} L}{\max_{H_A} L}$$

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Likelihood ratio test statistics

$$-2 \ln \Lambda$$

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Likelihood ratio test statistics

$$-2\ln\Lambda \underset{\text{app}}{\sim} \chi^2_\nu$$

Rao score test

Rao score (RS)

$$\mathbf{s}'(\widehat{\theta})\mathscr{F}^{-1}(\widehat{\theta})\mathbf{s}(\widehat{\theta})$$

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- $\mathbf{s}(\theta) = \left(\mathbf{s}_1'(\theta), \mathbf{s}_2'(\theta)\right)' = \left(\frac{\partial \ln L}{\partial \mathrm{vec}'\mu}, \frac{\partial \ln L}{\partial \mathrm{vech}'\Omega}\right)'$ – score vector

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- $\mathscr{F}(\theta) = -\mathrm{E}\left[\frac{\partial \mathbf{s}(\theta)}{\partial \theta'}\right]$ – Fisher information matrix (invertible)

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$$\mathbf{s}'(\widehat{\theta})\mathscr{F}^{-1}(\widehat{\theta})\mathbf{s}(\widehat{\theta}) \underset{\text{app}}{\sim} \chi^2_\nu$$

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RS test statistics

$$\begin{aligned} \mathrm{RS} &= \frac{nqp}{2} - \mathrm{tr}\left[(\widehat{\Psi}^{-1} \otimes \widehat{\Sigma}^{-1})\mathbf{X}\mathbf{Q}_{1_n}\mathbf{X}'\right] + \\ &+ \frac{1}{2n}\mathrm{tr}\left[(\widehat{\Psi}^{-1} \otimes \widehat{\Sigma}^{-1})\mathbf{X}\mathbf{Q}_{1_n}\mathbf{X}'(\widehat{\Psi}^{-1} \otimes \widehat{\Sigma}^{-1})\mathbf{X}\mathbf{Q}_{1_n}\mathbf{X}'\right] \end{aligned}$$

$$\mathbf{Q}_{1_n} = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$$

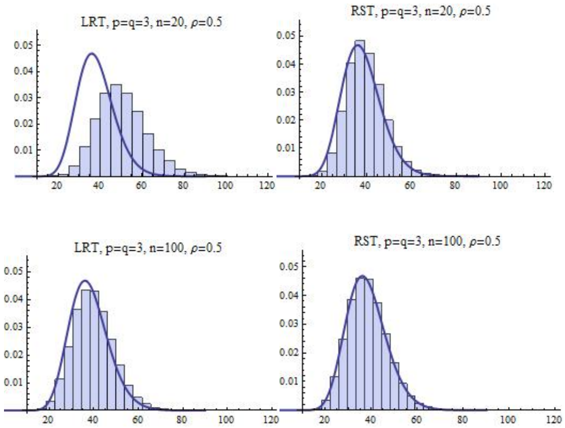
Simulation study, $q = 3$, $\alpha = 0.01$

$p \rightarrow$		3		4		5		7	
n	$\rho \downarrow$	E_α(LRT)	E_α(RST)	E_α(LRT)	E_α(RST)	E_α(LRT)	E_α(RST)	E_α(LRT)	E_α(RST)
10	-0.9	0.913	0.021	—	—	—	—	—	—
	-0.5	0.913	0.021	—	—	—	—	—	—
	-0.1	0.913	0.021	—	—	—	—	—	—
	0.1	0.913	0.021	—	—	—	—	—	—
	0.5	0.913	0.021	—	—	—	—	—	—
	0.9	0.913	0.021	—	—	—	—	—	—
15	-0.9	0.364	0.016	0.876	0.023	—	—	—	—
	-0.5	0.365	0.016	0.877	0.023	—	—	—	—
	-0.1	0.365	0.016	0.876	0.023	—	—	—	—
	0.1	0.365	0.016	0.876	0.023	—	—	—	—
	0.5	0.365	0.016	0.876	0.023	—	—	—	—
	0.9	0.365	0.015	0.876	0.022	—	—	—	—
20	-0.9	0.175	0.014	0.500	0.018	0.902	0.024	—	—
	-0.5	0.174	0.015	0.500	0.018	0.902	0.025	—	—
	-0.1	0.174	0.015	0.500	0.018	0.902	0.024	—	—
	0.1	0.174	0.015	0.500	0.018	0.901	0.024	—	—
	0.5	0.174	0.014	0.501	0.018	0.901	0.024	—	—
	0.9	0.174	0.014	0.500	0.019	0.901	0.024	—	—
25	-0.9	0.105	0.013	0.282	0.018	0.634	0.021	0.999	0.029
	-0.5	0.104	0.013	0.283	0.018	0.634	0.021	0.999	0.029
	-0.1	0.104	0.014	0.284	0.018	0.634	0.021	0.999	0.030
	0.1	0.104	0.013	0.284	0.018	0.633	0.021	0.999	0.029
	0.5	0.105	0.013	0.284	0.018	0.633	0.021	0.999	0.029
	0.9	0.105	0.014	0.283	0.017	0.634	0.021	0.999	0.029

Simulation study, $q = 3$, $\alpha = 0.01$

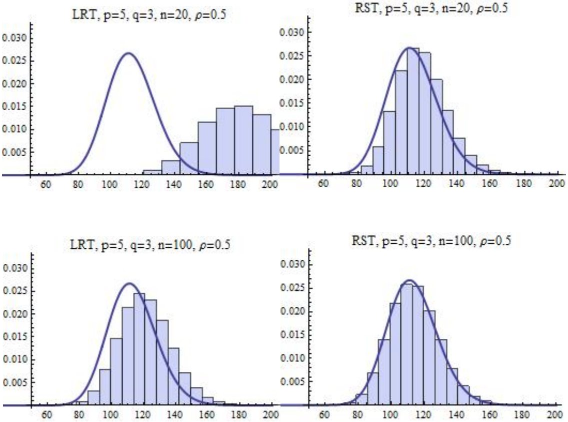
$p \rightarrow$		3		4		5		7	
n	$\rho \downarrow$	E_α(LRT)	E_α(RST)	E_α(LRT)	E_α(RST)	E_α(LRT)	E_α(RST)	E_α(LRT)	E_α(RST)
30	-0.9	0.075	0.012	0.182	0.016	0.420	0.019	0.965	0.025
	-0.5	0.075	0.012	0.182	0.016	0.420	0.019	0.965	0.025
	-0.1	0.075	0.012	0.182	0.016	0.420	0.019	0.965	0.025
	0.1	0.076	0.012	0.182	0.016	0.419	0.019	0.965	0.025
	0.5	0.076	0.013	0.182	0.016	0.420	0.019	0.965	0.025
	0.9	0.075	0.013	0.181	0.016	0.420	0.019	0.965	0.025
50	-0.9	0.035	0.011	0.063	0.013	0.127	0.015	0.445	0.017
	-0.5	0.035	0.011	0.063	0.013	0.127	0.015	0.445	0.017
	-0.1	0.035	0.011	0.063	0.013	0.126	0.015	0.445	0.018
	0.1	0.034	0.011	0.063	0.013	0.127	0.015	0.445	0.017
	0.5	0.034	0.011	0.063	0.013	0.126	0.015	0.445	0.017
	0.9	0.035	0.011	0.063	0.013	0.126	0.015	0.445	0.018
75	-0.9	0.023	0.011	0.035	0.012	0.060	0.014	0.172	0.014
	-0.5	0.024	0.012	0.035	0.012	0.059	0.014	0.172	0.014
	-0.1	0.024	0.011	0.035	0.012	0.060	0.014	0.172	0.014
	0.1	0.024	0.011	0.035	0.012	0.060	0.014	0.172	0.014
	0.5	0.024	0.011	0.035	0.012	0.060	0.014	0.172	0.014
	0.9	0.024	0.011	0.035	0.012	0.060	0.014	0.172	0.014
100	-0.9	0.018	0.010	0.026	0.012	0.039	0.013	0.096	0.014
	-0.5	0.019	0.011	0.026	0.011	0.039	0.013	0.096	0.014
	-0.1	0.019	0.011	0.026	0.012	0.040	0.013	0.096	0.014
	0.1	0.019	0.011	0.026	0.011	0.039	0.013	0.096	0.014
	0.5	0.019	0.011	0.025	0.012	0.039	0.013	0.096	0.014
	0.9	0.019	0.011	0.026	0.012	0.039	0.013	0.096	0.014

Simulation study, $p = q = 3$



Plots of the empirical histogram and its limiting χ^2 distribution for LRT and RST statistics for $n = 20$ and $n = 100$

Simulation study, $p = 5$, $q = 3$



Plots of the empirical histogram and its limiting χ^2 distribution for LRT and RST statistics for $n = 20$ and $n = 100$

Simulation study, $p = q = 3$

n	$Q_{\mathrm{LRT}}(90)$	$Q_{\mathrm{LRT}}(95)$	$Q_{\mathrm{LRT}}(99)$	$Q_{\mathrm{RST}}(90)$	$Q_{\mathrm{RST}}(95)$	$Q_{\mathrm{RST}}(99)$
4	—	—	—	58.191	62.252	72.835
6	—	—	—	54.549	58.689	67.359
8	—	—	—	53.294	57.294	66.054
10	130.369	145.192	178.540	52.291	56.225	64.950
15	75.229	81.465	93.813	51.237	55.112	63.284
20	66.041	71.300	81.863	50.711	54.595	62.906
30	59.122	63.816	73.082	50.392	54.368	62.356
50	54.710	59.064	67.374	49.917	53.768	61.587
75	52.960	57.158	65.284	49.971	53.775	61.702
100	51.873	55.930	64.332	49.601	53.553	61.518
150	51.174	55.041	63.101	49.741	53.525	61.358
200	50.611	54.499	62.215	49.562	53.388	60.868
∞	49.513	53.384	61.162	49.513	53.384	61.162

Empirical 90th, 95th and 99th percentiles of the null distribution of LRT and RST statistics based on 50000 simulations

Simulation study, $p = 15$, $q = 3$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	1425.192	1457.032	1525.867
6	—	—	—	1292.755	1319.259	1377.402
8	—	—	—	1234.590	1258.540	1308.822
10	—	—	—	1202.040	1224.829	1273.590
15	—	—	—	1160.494	1180.659	1223.119
20	—	—	—	1141.722	1161.032	1199.170
30	—	—	—	1122.316	1141.628	1180.741
50	1922.310	1956.998	2026.270	1107.564	1125.782	1160.917
75	1445.318	1468.808	1514.623	1100.496	1118.716	1152.721
100	1321.738	1342.494	1384.453	1097.159	1115.501	1148.509
150	1227.116	1246.645	1282.713	1093.892	1111.193	1143.346
200	1187.107	1205.345	1242.713	1091.588	1109.742	1143.240
∞	1086.521	1103.702	1136.416	1086.521	1103.702	1136.416

Empirical 90th, 95th and 99th percentiles of the null distribution of LRT and RST statistics based on 50000 simulations

Simulation study, $q = 3$

$p \rightarrow$ $n \downarrow$	3	4	5	7	10	15
4	17.528 —	21.586 —	24.111 —	27.281 —	29.712 —	31.170 —
10	5.611 *163.305*	7.209 —	8.200 —	9.485 —	10.175 —	10.632 —
30	1.776 *19.409*	2.305 *24.561*	2.672 *31.288*	2.920 *49.209*	3.219 —	3.294 —
50	0.816 *10.497*	1.124 *12.779*	1.577 *15.825*	1.604 *22.052*	1.889 *34.843*	1.937 *76.923*
75	0.926 *6.963*	0.879 *8.196*	1.029 *9.875*	1.203 *13.175*	1.267 *19.485*	1.286 *33.023*
100	0.179 *4.767*	0.586 *5.786*	0.725 *7.068*	1.001 *9.541*	0.946 *13.611*	0.979 *21.649*

Percent errors between the RST statistics and their ENDs for 90th percentile for different values of p and n

Example - Dental data (Timm, 1980)

$n = 9$ subjects
$q = 3$ measurements
$p = 3$ time points

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$n = 9$ subjects
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q	ν	LRT (END p–value)	RST (END p–value)	LRT (χ^2_ν p–value)	RST (χ^2_ν p–value)
1,2,3	38	—	57.0597	—	57.0597
			$((0.01; 0.05))$		(0.024)
1,2	17	22.8021	15.6992	22.8021	15.6992
		(> 0.10)	(> 0.10)	(0.156)	(0.455)
1,3	17	38.4535	22.8755	38.4535	22.8755
		(> 0.10)	(> 0.10)	(0.002)	(0.153)

Calculated values of LRT, RST statistics and their p-values along with the p-values of the limiting χ^2 distribution

Percentiles, $p = 3$, $q = 3$, $df = 38$

$$\mathrm{RST} = 57.0597$$

n	$Q_{\mathrm{LRT}}(90)$	$Q_{\mathrm{LRT}}(95)$	$Q_{\mathrm{LRT}}(99)$	$Q_{\mathrm{RST}}(90)$	$Q_{\mathrm{RST}}(95)$	$Q_{\mathrm{RST}}(99)$
4	—	—	—	58.191	62.252	72.835
5	—	—	—	56.030	60.049	69.158
6	—	—	—	54.549	58.689	67.359
7	—	—	—	53.908	57.968	67.074
8	—	—	—	53.294	57.294	66.054
9	—	—	—	52.531	56.505	65.288
10	130.369	145,192	178.540	52.291	56.225	64.950
100	51.873	55.930	64.332	49.601	53.553	61.518
∞	49.513	53.384	61.162	49.513	53.384	61.162

Percentiles, $p = 3$, $q = 3$, $df = 38$

$$\text{RST} = 57.0597$$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	58.191	62.252	72.835
5	—	—	—	56.030	60.049	69.158
6	—	—	—	54.549	58.689	67.359
7	—	—	—	53.908	57.968	67.074
8	—	—	—	53.294	57.294	66.054
9	—	—	—	52.531	56.505	65.288
10	130.369	145,192	178.540	52.291	56.225	64.950
100	51.873	55.930	64.332	49.601	53.553	61.518
∞	49.513	53.384	61.162	49.513	53.384	61.162

$$0.01 < \text{END } p\text{-value} < 0.05$$

Percentiles, $p = 3$, $q = 3$, $df = 38$

$$\mathrm{RST} = 57.0597$$

n	$Q_{\mathrm{LRT}}(90)$	$Q_{\mathrm{LRT}}(95)$	$Q_{\mathrm{LRT}}(99)$	$Q_{\mathrm{RST}}(90)$	$Q_{\mathrm{RST}}(95)$	$Q_{\mathrm{RST}}(99)$
4	—	—	—	58.191	62.252	72.835
5	—	—	—	56.030	60.049	69.158
6	—	—	—	54.549	58.689	67.359
7	—	—	—	53.908	57.968	67.074
8	—	—	—	53.294	57.294	66.054
9	—	—	—	52.531	56.505	65.288
10	130.369	145,192	178.540	52.291	56.225	64.950
100	51.873	55.930	64.332	49.601	53.553	61.518
∞	49.513	53.384	61.162	49.513	53.384	61.162

$0.01 <$ END p-value < 0.05

$0.01 < \chi^2_{38}$ p-value $= 0.024 > 0.05$

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\text{LRT} = 22.8021$$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\text{LRT} = 22.8021$$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

END p-value > 0.10

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\text{LRT} = 22.8021$$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

END p-value > 0.10

χ^2_{17} p-value $= 0.156 > 0.10$

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\mathrm{RST} = 15.6992$$

n	$Q_{\mathrm{LRT}}(90)$	$Q_{\mathrm{LRT}}(95)$	$Q_{\mathrm{LRT}}(99)$	$Q_{\mathrm{RST}}(90)$	$Q_{\mathrm{RST}}(95)$	$Q_{\mathrm{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\mathrm{RST} = 15.6992$$

n	$Q_{\mathrm{LRT}}(90)$	$Q_{\mathrm{LRT}}(95)$	$Q_{\mathrm{LRT}}(99)$	$Q_{\mathrm{RST}}(90)$	$Q_{\mathrm{RST}}(95)$	$Q_{\mathrm{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

END p-value > 0.10

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\text{RST} = 15.6992$$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

END p-value > 0.10

χ^2_{17} p-value $= 0.455 > 0.10$

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\text{LRT} = 38.4535$$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\text{LRT} = 38.4535$$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

END p-value > 0.10

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\text{LRT} = 38.4535$$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

END p-value > 0.10

χ^2_{17} p-value $= 0.002 < 0.01$

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\text{RST} = 22.8755$$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\text{RST} = 22.8755$$

n	$Q_{\text{LRT}}(90)$	$Q_{\text{LRT}}(95)$	$Q_{\text{LRT}}(99)$	$Q_{\text{RST}}(90)$	$Q_{\text{RST}}(95)$	$Q_{\text{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

END p-value > 0.10

Percentiles, $p = 3$, $q = 2$, $df = 17$

$$\mathrm{RST} = 22.8755$$

n	$Q_{\mathrm{LRT}}(90)$	$Q_{\mathrm{LRT}}(95)$	$Q_{\mathrm{LRT}}(99)$	$Q_{\mathrm{RST}}(90)$	$Q_{\mathrm{RST}}(95)$	$Q_{\mathrm{RST}}(99)$
4	—	—	—	28.157	31.330	37.403
5	—	—	—	27.196	30.270	36.913
6	—	—	—	26.739	29.543	36.111
7	68.892	79.123	101.522	26.366	29.321	36.104
8	51.295	58.139	72.004	26.144	29.045	35.302
9	44.406	50.089	61.554	25.856	28.661	35.251
10	40.762	45.806	55.971	25.680	28.566	34.701
100	25.610	28.532	34.406	24.823	27.554	33.336
∞	24.769	27.587	33.409	24.769	27.587	33.409

END p-value > 0.10

χ^2_{17} p-value $= 0.153 > 0.10$

Conclusions

- RST statistic depends only on the data matrix $\boldsymbol{X}$, an estimate of Ψ with an explicit expression in ρ, and an estimate of the $q \times q$ dimensional variance-covariance matrix Σ. Thus, the minimum number of samples needed to calculate RST statistic is only $q+1$, which does not even depend on p, whereas the minimum number of samples needed to calculate LRT statistic is $pq+1$, which can grow very fast with the increase in p. Thus, the RST is an immense improvement over the LRT: one can test the null hypothesis H_0 with only $q+1$ samples using RST.

Conclusions

- From the Tables, as well as from Figures, we observe that for small samples as well as for moderate samples END of RST statistic is much more closer to the limiting χ^2 distribution than its counterpart LRT statistic. Thus, we conclude that END of RST statistic performs much better than END of LRT statistic for both small and moderate sample studies, and it is then prudent to use END of RST statistic as opposed to END of LRT statistic for any real life application.

Conclusions

- From the examples we see that the inference can changes if we use END as opposed to the limiting χ^2 distribution which is very conservative, especially if the test statistic value lies in the close neighborhood of the critical value of the χ^2 distribution. However, we see that the conclusions drawn from END using RST and the limiting χ^2 distribution are the same all. These observations suggest us to use RST instead of LRT for testing separability of the variance-covariance matrix for small and moderate sample sizes, and especially in small sample sizes. If END of RST is available, use END for precise conclusion, otherwise calculate RST statistic and just use the limiting χ^2 distribution, the decision would not differ much.

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Simulations – power

n	LRT	RST
4	—	0.064
6	—	0.217
8	—	0.414
10	0.417	0.684
15	1.000	0.999
20	1.000	1.000
25	1.000	1.000
30	1.000	1.000
50	1.000	1.000
75	1.000	1.000
100	1.000	1.000

Empirical power of LRT and RST for variuous n and $p = q = 3$ for $\alpha = 0.01$ based on 50,000 simulations (nonseparable alternative)

Simulations – power

n	LRT	RST
4	—	0.010
6	—	0.010
8	—	0.010
10	0.010	0.010
15	0.011	0.011
20	0.011	0.011
25	0.013	0.012
30	0.013	0.013
50	0.018	0.016
75	0.024	0.022
100	0.029	0.027

Empirical power of LRT and RST for variuous n and $p = q = 3$ for $\alpha = 0.01$ based on 50,000 simulations (separable alternative, close to H_0)