On some applications of saddle-point matrices

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Introduction

In standard notation $\mathbf{e} \in \mathbb{R}^n$ is the column vector with all components of ones, and $\mathbf{E} \in \mathbb{R}^{n\times n}$ stands for the matrix with all entries equal to one. Therefore $\mathbf{E} = \mathbf{e}\mathbf{e}^T$. For a given square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ denote by $(\mathbf{A}, \mathbf{e})$ the matrix, often called the saddle point matrix, and defined as follows

$$(\mathbf{A}, \mathbf{e}) = \begin{bmatrix} \mathbf{A} & \mathbf{e} \\ \mathbf{e}^T & 0 \end{bmatrix} \quad (1)$$

Introduction

If the matrix $\mathbf{A}$ is symmetric, then $(\mathbf{A}, \mathbf{e})$ may be interpreted as the bordered Hessian of a standard quadratic program over the standard simplex, and it is called the Karush-Kuhn-Tucker matrix of the program, which is known to have a large spectrum of applications (for a review see, for instance, Bomze (1998)).

Proposition. *For any real numbers α, β, the determinant of the matrix $\alpha\mathbf{A} + \beta\mathbf{E}$ can be expressed as follows:*

$$det(\alpha\mathbf{A} + \beta\mathbf{E}) = \alpha^{n-1}[\alpha det(\mathbf{A}) - \beta det(\mathbf{A}, \mathbf{e})] \qquad (2)$$

Corollary 1.

a) From the equality (2), putting $\alpha = \beta = 1$, we immediately obtain that the determinant of $(\mathbf{A}, \mathbf{e})$ depends on the determinants of the matrices $\mathbf{A}$ and $\mathbf{A} + \mathbf{E}$, and the relation is expressed by the formula:

$$det(\mathbf{A}, \mathbf{e}) = det(\mathbf{A}) - det(\mathbf{A} + \mathbf{E}) \tag{3}$$

b) Assume that $(\mathbf{A}, \mathbf{e})$ is nonsingular and let $\alpha = 1$, $\beta = \beta_0 = \dfrac{det(\mathbf{A})}{det(\mathbf{A}, \mathbf{e})}$. Then the matrix of the form $det(\mathbf{A} + \beta_0 \mathbf{E})$ is singular.

Introduction

Corollary 2.
It is easy to see that equality (2) also implies, that a matrix given as a linear combination of the form $det(\mathbf{A}, \mathbf{e})\mathbf{A} + det(\mathbf{A})\mathbf{E}$ is singular for any matrix $\mathbf{A}$.

For more properties of saddle point matrices see, for example M. Benzi, G. H. Golub, and J. Liesen(2005).

Value of the game

A bimatrix game is the game described by an ordered pair of payoff matrices $\mathbf{A} = [a_{ij]}$ and $\mathbf{B} = [b_{ij}]$, with equal dimensions. When the row and column players choose their i-th and j-th pure strategies, respectively, the row player's payoff is a_{ij} and the column player's payoff is b_{ij}. A mixed strategy is a probability vector $\mathbf{x}$ specifying the probability with which each pure strategy is played.

Value of the game

If these probabilities are all positive, the vector $\mathbf{x}$ is said to be completely mixed. A completely mixed Nash equilibrium is one in which both players' strategies are completely mixed. This paper concerns games in which the number of pure strategies is the same for both players, so that $\mathbf{A}$ and $\mathbf{B}$ are square, $n \times n$ matrices $(n \geqslant 2)$.

Value of the game

Theorem. For any $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$, where $det(\mathbf{B}, \mathbf{e}) \neq 0$, the bimatrix game $[\mathbf{A}, \mathbf{B}]$ in which players strategies are completely mixed, has the players equilibrium payoffs $\nu(\mathbf{A}), \nu(\mathbf{B})$ equal to

$$\nu(\mathbf{A}) = -\frac{det(\mathbf{A})}{det(\mathbf{A}, \mathbf{e})}, \quad \nu(\mathbf{B}) = -\frac{det(\mathbf{B})}{det(\mathbf{A}, \mathbf{e})} \qquad (4)$$

Mean Value Theorem

Let a real-valued function f be continuous on a closed interval $< a, b >$ and differentiable on the open interval (a, b). The well-known Rolle's Theorem states:

if $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$ (the graph of f has a horizontal tangent somewhere between a and b).

P r o o f.
Let $F(x)$ be the function of the form

$$F(x) = det(\mathbf{A}, \mathbf{e}), \; \mathbf{A} = \begin{bmatrix} x & f(x) & 0 \\ a & f(a) & 0 \\ b & f(b) & 0 \end{bmatrix}. \qquad (5)$$

Mean Value Theorem

Function $F(x)$ is also continuous and differentiable on the open interval (a, b). It is easy to see that $F(a) = F(b) = 0$. Since then we are in a position to apply Rolle's Theorem. Therefore the derivative $F'(x) = 0$, and we have

$$F'(x) = det'(\mathbf{A}, \mathbf{e}) = det \begin{bmatrix} 1 & f'(x) & 0 & 1 \\ a & f(a) & 0 & 1 \\ b & f(b) & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = 0. \qquad (6)$$

So, there exists at least one point $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ and the proof of the Mean Value Theorem is complete.

Remark.

The proof of the Mean Value Theorem (a more descriptive name would be Average Slope Theorem), which generalizes Rolle's Theorem, is accomplished by finding a way to apply Rolle's Theorem. The above is a non-standard proof of a standard formulation of the Mean Value Theorem.

Another non-standard proof you can find in R. Almeida (2008) (with analysis technics) or in H. Diener and I. Loeb (2011) (as an extending a constructive reverse perspective).

Area of a triangle

Hero (or Heron) of Alexandria (c. 10 – 70 AD) in the book Metrica described how to calculate surfaces and volumes of diverse objects – among them the area of a triangle via lenghs of its sides. It is known as Heron's formula (also as Hero's formula) and plays an important theorem in plane geometry. Heron's formula is a special case of Brahmagupta's formula for the area of a cyclic quadrilateral. Heron's formula and Brahmagupta's formula are both special cases of Bretschneider's formula for the area of a quadrilateral.

Volume of a tetrahedron

Heron's formula can be obtained from Brahmagupta's formula or Bretschneider's formula by setting one of the sides of the quadrilateral to zero. Brahmagupta (597–668 CE) was an Indian mathematician and astronomer who wrote two important works on mathematics and astronomy: the Brahmasphutasiddhanta (Extensive Treatise of Brahma) (628), a theoretical treatise, and the Khandakhadyaka, a more practical text.

Volume of a tetrahedron

Carl Anton Bretschneider (1808 – 1878) was a mathematician from Gotha, Germany. Bretschneider worked in different ares of mathematics, among them in geometry. Expressing Heron's formula with a Cayley–Menger determinant in terms of the squares of the distances between the three given vertices, illustrates its similarity to Tartaglia's formula for the volume of a three-simplex.

$$S = \frac{1}{4}(-det \begin{bmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix})^{1/2}$$

Volume of a tetrahedron

Niccolo Fontana Tartaglia (about 1500 – 1557) was an Italian mathematician and engineer, who proposed a formula for the volume of a tetrahedron (including any irregular tetrahedra) as the Cayley–Menger determinant of the distance values measured pairwise between its four corners:

$$V^2 = \frac{1}{288} det \begin{bmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & 1 \\ d_{12}^2 & 0 & d_{23}^2 & d_{24}^2 & 1 \\ d_{13}^2 & d_{23}^2 & 0 & d_{34}^2 & 1 \\ d_{14}^2 & d_{24} & d_{34} & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Volume of an n-dimensional hypertethahedron

The proposal for the *volume* of an n-dimensional figure – *hypertethahedron*

$$M_n = \frac{1}{n!2^{n/2}} \mid det(\mathbf{A}, \mathbf{e}) \mid^{1/2}$$

The measure coincides with the formula proposed by M. Griffiths (2005) for regular hypertethahedrons with all sides equal to 1.

Volume of an n-dimensional hypertethahedron

n = 1
(1-dimensional tetrahedron = a segment)

Length of the segment :

$$M_1 = \frac{1}{2^{1/2}} \mid det(\mathbf{A}, \mathbf{e}) \mid^{1/2}; \; \mathbf{A} = \begin{bmatrix} 0 & a^2 \\ a^2 & 0 \end{bmatrix}$$

Volume of an n-dimensional hypertethahedron

n = 2
(2-dimensional tetrahedron = a triangle; a, b, c are edges)

Area of the triangle:

$$M_2 = \frac{1}{4} \mid det(\mathbf{A}, \mathbf{e}) \mid^{1/2}; \ \mathbf{A} = \begin{bmatrix} 0 & a^2 & b^2 \\ a^2 & 0 & c^2 \\ b^2 & c^2 & 0 \end{bmatrix}$$

Volume of an n-dimensional hypertethahedron

n = 3
(3-dimensional tetrahedron = a pyramid with the triangle base a, b, c and other edges d, e, f connecting base with the top)

Volume of tetrahedron:

$$M_3 = \frac{1}{4} \mid det(\mathbf{A}, \mathbf{e}) \mid^{1/2}; \; \mathbf{A} = \begin{bmatrix} 0 & a^2 & b^2 & d^2 \\ a^2 & 0 & c^2 & e^2 \\ b^2 & c^2 & 0 & f^2 \\ d^2 & e^2 & f^2 & 0 \end{bmatrix}$$

Volume of an n-dimensional hypertethahedron

0	1	1
1	0	1
1	1	0

det	2
V	1

0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

det	-3	V	0,43

0	1	1	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

det	4	V	0,12

0	1	1	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	0	1	1
1	1	1	1	0	1
1	1	1	1	1	0

det	-5	V	0,02

Volume of an n-dimensional hypertethahedron

0	2	1
2	0	1
1	1	0

det	4
V	1,41

0	2	2	1
2	0	2	1
2	2	0	1
1	1	1	0

det	-12	V	0,87

0	2	2	2	1
2	0	2	2	1
2	2	0	2	1
2	2	2	0	1
1	1	1	1	0

det	32	V	0,33

0	2	2	2	2	1
2	0	2	2	2	1
2	2	0	2	2	1
2	2	2	0	2	1
2	2	2	2	0	1
1	1	1	1	1	0

det	-80	V	0,09

Volume of an n-dimensional hypertethahedron

0	3	1
3	0	1
1	1	0

det	6
V	1,73

0	3	3	1
3	0	3	1
3	3	0	1
1	1	1	0

det	-27	V	1,3

0	3	3	3	1
3	0	3	3	1
3	3	0	3	1
3	3	3	0	1
1	1	1	1	0

det	108	V	0,61

0	3	3	3	3	1
3	0	3	3	3	1
3	3	0	3	3	1
3	3	3	0	3	1
3	3	3	3	0	1
1	1	1	1	1	0

det	-405	V	0,21

Volume of an n-dimensional hypertethahedron

0	3	1
3	0	1
1	1	0

det 6

V	1,73

0	3	4	1
3	0	5	1
4	5	0	1
1	1	1	0

det	-44	V	1,66

0	3	4	3	1
3	0	5	3	1
4	5	0	3	1
5	3	3	0	1
1	1	1	1	0

det	164	V	0,75

0	3	4	3	3	1
3	0	5	3	3	1
4	5	0	3	3	1
5	3	3	0	3	1
3	3	3	3	0	1
1	1	1	1	1	0

det	-528	V	0,24

Volume of an n-dimensional hypertethahedron

0	9	16	36	45	1
9	0	25	42	37	1
16	25	0	38	40	1
36	42	38	0	43	1
45	37	40	43	0	1
1	1	1	1	1	0

det	-1876753	V	14,27

0	22	27	36	45	1
22	0	25	42	37	1
27	25	0	38	40	1
36	42	38	0	43	1
45	37	40	43	0	1
1	1	1	1	1	0

det	-6048044	V	25,62

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