Explicit Estimators for a Banded Covariance Matrix in a Multivariate Normal Distribution

Presentation

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History

- Patterned covariance matrices
- Banded covariance matrices
- Methods: explicit, maximum likelihood and back again
List of symbols

\( A_{m,n} \) - matrix of size \( m \times n \)
\( M_{m,n} \) - the set of all matrices of size \( m \times n \)
\( a_{ij} \) - matrix element of the \( i \)-th row and \( j \)-th column
\( a_n \) - vector of size \( n \)
\( c \) - scalar
\( X \) - random matrix
\( x \) - random vector
\( X \) - random variable
Explicit Estimator

Previous results

**Proposition 1**

Let $X \sim N_{p,n}(\mu 1_n', \Sigma^{(m)}_{(p)}, I_n)$. Explicit estimators are given by

\[ \hat{\mu}_i = \frac{1}{n} x_i' 1_n, \]

\[ \hat{\sigma}_{i,i} = \frac{1}{n} x_i' C x_i \text{ for } i = 1, \ldots, p, \]

\[ \hat{\sigma}_{i,i+1} = \frac{1}{n} \hat{r}_i' C x_{i+1} \text{ for } i = 1, \ldots, p - 1, \]

where $\hat{r}_1 = y_1$ and $\hat{r}_i = x_i - \hat{s}_i \hat{r}_{i-1}$ for $i = 2, \ldots, p - 1$,

\[ \hat{s}_i = \frac{\hat{r}_{i-1}' C x_i}{\hat{r}_{i-1}' C x_{i-1}}, \]

where $C = I_n - \frac{1}{n} 1_n 1_n'$. 

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Explicit Estimators for a Banded Covariance Matrix
Previous results

Theorem 1

The estimator \( \hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_p)' \) given in Proposition 1 is unbiased and consistent, and the estimator \( \hat{\Sigma}^{(m)}_{(p)} = (\hat{\sigma}_{ij}) \) is consistent.
Explicit Estimator  
Purpose of this work  

Goals:  
- Find an unbiased estimator for the covariance matrix.  
- Generalize results into a general linear model.  

Limitations:  
- Study the case where $\Sigma^{(p)}_{(1)}$ instead of $\Sigma^{(p)}_{(m)}$.  

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Find an unbiased estimator
Rewriting of estimator

Proposition 2

Let \( X \sim N_{p,n}(\mu 1_n', \Sigma^{(1)}_{(p)}, I_n) \). Explicit estimators are given by

\[
\hat{\sigma}_{i,i+1} = \frac{1}{n} x_i' A_{i-1} x_{i+1} \text{ for } i = 1, \ldots, p - 1,
\]

where \( A_i = C - C \hat{r}_i (\hat{r}_i'C\hat{r}_i)^{-1} \hat{r}_i'C \),

with \( A_0 = C \),

where \( \hat{r}_1 = y_1 \) and \( \hat{r}_i = x_i - \frac{\hat{r}_{i-1}'Cx_i}{\hat{r}_{i-1}'Cx_{i-1}} \hat{r}_{i-1} \) for \( i = 2, \ldots, p - 1 \),

with \( C = I_n - \frac{1}{n} n 1_n 1_n' \).
Definition 1

Let \( x \sim N_n(\mu_x, I_n), y \sim N_n(\mu_y, I_n) \) and \( A \in M_{n,n} \). Then \( x' Ay \) is called a bilinear form.

Theorem 2

The bilinear form \( x' Ay \) has the following properties.

(i) \( E[x' Ay] = \text{tr}(A \text{cov}(x, y)) \)

(ii) \( \text{var}[x' Ay] = \text{tr}(A \text{cov}(x, y))^2 + \text{tr}(A \text{var}(x)A \text{var}(y)) = \text{tr}(A) \text{cov}(x, y)^2 + \text{tr}(A^2) \text{var}(x) \text{var}(y) \).
Find an unbiased estimator
Properties of the central matrix

The central matrix for \( \hat{\sigma}_{i,i+1} = \frac{1}{n} x'_i A_{i-1} x_{i+1} \)

\[
A_i = C - C \hat{r}_i (\hat{r}_i' C \hat{r}_i)^{-1} \hat{r}_i' C
\]

Properties:
- Idempotent, \( A_i^2 = A_i \)
- Symmetric, \( A_i' = A_i \)

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Proposition 3

Let $X \sim N_{p,n}(\mu 1_n', \Sigma^{(1)}_{(p)}, I_n)$. Explicit estimators are given by

$$\hat{\sigma}_{ii} = \frac{1}{n-1} x_i' C x_i \text{ for } i = 1, \ldots, p,$$

$$\hat{\sigma}_{12} = \frac{1}{n-1} x_1' C x_2,$$

$$\hat{\sigma}_{i,i+1} = \frac{1}{n-2} x_i' A_{i-1} x_{i+1} \text{ for } i = 2, \ldots, p-1,$$

where $A_i = C - C \hat{r}_i (\hat{r}_i' C \hat{r}_i)^{-1} \hat{r}_i' C$, with $A_0 = C$,

where $\hat{r}_1 = y_1$ and $\hat{r}_i = x_i - \frac{\hat{r}_{i-1}' C x_i}{\hat{r}_{i-1}' C x_{i-1}} \hat{r}_{i-1}$ for $i = 2, \ldots, p-1$,

with $C = I_n - \frac{1}{n} 1_n 1_n'$.
Find an unbiased estimator

Results

Theorem 3

The estimators from Proposition 3 are unbiased and consistent.

Variance is known:

\[ \text{var}(\hat{\sigma}_{i,i+1}) = \frac{\sigma_{i,i+1}^2 + \sigma_{ii} \sigma_{i+1,i+1}}{n-2} \]
Generalization to a general linear model

General linear model

Assumptions:

General linear model

\[ Y = XB + E \sim N_{n,p}(XB, I_n, \Sigma^{(p)}_{(1)}) \]

- \( Y \) and \( E \) are \( n \times m \) random matrices
- \( X \) is a known \( n \times p \)-design matrix with full rank
- \( B \) is an unknown \( p \times m \)-matrix of regression coefficients.
- \( n \geq m + p \), and the rows of the error matrix \( E \) are independent \( N_m(0, \Sigma) \) random vectors.
Problems when generalizing

- The expected value $XB$ differs from $\mu 1'$.
- The design matrix affects the degrees of freedom.
Transformation:

\[(Y - XB) \sim N(0, I_n, \Sigma) \text{ can be treated as}\]

\[(y_i - Xb_i) \sim N(0, \Sigma) \text{ were each part is handled separately}\]
Let \( Y = XB \sim N_{p,n}(XB, \Sigma^{(1)}_{(p)}, I_n) \), where \( \text{rank}(X) = k \). Explicit estimators are given by

\[
\hat{B} = (X'X)^{-1}X'Y,
\]

\[
\hat{\sigma}_{ii} = \frac{1}{n-k}y'_iDy_i \text{ for } i = 1, \ldots, p,
\]

\[
\hat{\sigma}_{i,i+1} = \frac{1}{n-k-1}y'_iA_{i-1}x_{i+1} \text{ for } i = 2, \ldots, p-1,
\]

where \( A_i = D - D\hat{r}_i(\hat{r}'_iD\hat{r}_i)^{-1}\hat{r}'_iD \) with \( A_0 = D \),

where \( \hat{r}_1 = y_1 \) and \( \hat{r}_i = y_i - \frac{\hat{r}'_{i-1}Dy_i}{\hat{r}'_{i-1}Dy_{i-1}}\hat{r}_{i-1} \) for \( i = 2, \ldots, p-1 \),

with \( D = I_n - X(X'X)^{-1}X' \).
Theorem 4

*The estimators from Proposition 4 are unbiased and consistent.*

Known variance:

\[
\text{var}(\hat{\sigma}_{i,i+1}) = \frac{1}{n-2}(A)(\sigma_{i,i+1}^2 + \sigma_{i,i} \sigma_{i+1,i+1})
\]
Based on the 100000 averages of samples with $n=20$, explicit unbiased average estimators, with true value within parenthesis, are given by,

$$
\hat{\Sigma}_{\text{new}} = \begin{pmatrix}
4.99501(5) & 1.99590(2) & 0.00000 & 0.00000 \\
1.99590(2) & 4.99238(5) & 0.99678(1) & 0.00000 \\
0.00000 & 0.99678(1) & 5.00026(5) & 3.00265(3) \\
0.00000 & 0.00000 & 3.00265(3) & 5.00368(5)
\end{pmatrix},
$$

and the previous estimators are given by,

$$
\hat{\Sigma}_{\text{prev}} = \begin{pmatrix}
4.74526(5) & 1.89611(2) & 0.00000 & 0.00000 \\
1.89611(2) & 4.74276(5) & 0.89710(1) & 0.00000 \\
0.00000 & 0.89710(1) & 4.75025(5) & 2.70239(3) \\
0.00000 & 0.00000 & 2.70239(3) & 4.75350(5)
\end{pmatrix}.
$$
Based on the 100000 averages of samples with $n=80$ and 20 regression parameters, where $X$ and $B$ were randomly generated, unbiased explicit average estimators, with true value within parenthesis, are given by,

\[
\hat{\Sigma}_{\text{new}} = \begin{pmatrix}
3.9986(4) & 0.9997(1) & 0 & 0 & 0 \\
0.9997(1) & 3.0051(3) & 2.0024(2) & 0 & 0 \\
0 & 2.0024(2) & 4.9989(5) & 2.9976(3) & 0 \\
0 & 0 & 2.9976(3) & 4.9941(5) & 2.9950(3) \\
0 & 0 & 0 & 2.9950(3) & 4.9935(5)
\end{pmatrix},
\]

and the previous estimators are given by,

\[
\hat{\Sigma}_{\text{prev}} = \begin{pmatrix}
2.9989(4) & 0.7497(2) & 0 & 0 & 0 \\
0.7497(2) & 2.2538(3) & 1.4768(2) & 0 & 0 \\
0 & 1.4768(2) & 3.7492(5) & 2.2107(3) & 0 \\
0 & 0 & 2.2107(3) & 3.7455(5) & 2.2088(3) \\
0 & 0 & 0 & 2.2088(3) & 3.7451(5)
\end{pmatrix}.
\]
Conclusion:

- The unbiased version makes an considerable improvement
Topics:

- Find unbiased estimator for $\Sigma_{(m)}^{(p)}$
- Compare it to other estimators (for example MLE) for banded matrices.
- Study the variance to determine efficiency