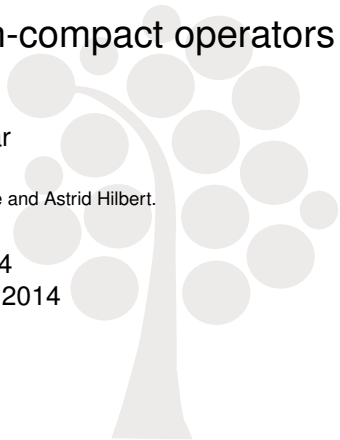


# On the functional Hodrick-Prescott filter with compact operators and non-compact operators

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The univariate HP filter extracts a 'signal'  $y(\alpha, x) = (y_1(\alpha, x), \dots, y_T(\alpha, x))$  from a noisy time series  $x = (x_1, \dots, x_T)$  as a minimizer of

$$\sum_{t=1}^T (x_t - y_t)^2 + \alpha \sum_{t=3}^T (y_t - 2y_{t-1} + y_{t-2})^2, \quad (1)$$

with respect to  $y = (y_1, \dots, y_T)$ , for an appropriately chosen positive parameter  $\alpha$ , called the smoothing parameter.

The second order differencing operator  $Py(t) = y_t - 2y_{t-1} + y_{t-2}$  is written in vector form as the following  $(T-2) \times T$ -matrix

$$P := \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & -2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

To determine an appropriate value of the smoothing parameter  $\alpha$ , Hodrick and Prescott (1997) suggest the time series  $(x, y)$  satisfies the following linear mixed model:

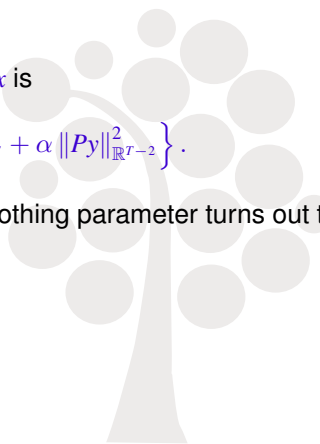
$$\begin{cases} x = y + u, \\ Py = v. \end{cases} \quad (2)$$

where,  $u \sim N(0, \sigma_u^2 I_T)$  and  $v \sim N(0, \sigma_v^2 I_{T-2})$ .

The 'optimal smooth' signal associated with  $x$  is

$$\bar{y}(\alpha, x) := \arg \min_y \left\{ \|x - y\|_{\mathbb{R}^T}^2 + \alpha \|Py\|_{\mathbb{R}^{T-2}}^2 \right\}. \quad (3)$$

Using the model above, the appropriate smoothing parameter turns out to be the noise-to-signal ratio  $\alpha^* = \sigma_u^2 / \sigma_v^2$ .



Schlicht in (2005) proved that the noise-to-signal ratio satisfies

$$E[y|x] = y\left(\frac{\sigma_u^2}{\sigma_v^2}, x\right), \quad (4)$$

where  $E[y|x]$  is the best predictor of any signal  $y$  given the time series  $x$ . Dermoune *et al.* proposed in (2009) an optimality criterion for choosing the smoothing parameter for the HP-filter. The smoothing parameter  $\alpha$  is chosen as the following:

$$\alpha^* = \arg \min_{\alpha} \{ \|E[y|x] - y(\alpha, x)\|^2 \} \quad (5)$$

Furthermore, Dermoune *et al.* (2009) proposed a multivariate version of the HP filter and determined the possible optimal smoothing parameters.

## Definition

Let  $H_1$  and  $H_2$  be two separable Hilbert spaces, with norms  $\|\cdot\|_{H_i}$  and inner products  $\langle \cdot, \cdot \rangle_{H_i}$ ,  $i = 1, 2$ , and  $x \in H_1$  be a functional time series of observables. A functional Hodrick-Prescott filter reconstructs an 'optimal smooth signal'  $y \in H_1$  that solves an equation  $Ay = v$ , corrupted by a noise  $v$  which is a priori unobservable, from observations  $x$  corrupted by a noise  $u$  which is also a priori unobservable:

$$\begin{cases} x = y + u, \\ Ay = v, \end{cases} \quad (6)$$

given the linear operator  $A : H_1 \rightarrow H_2$  and  $u, v$  are independent random variables with zero mean and covariance operators  $\Sigma_u$  and  $\Sigma_v$  respectively.

The 'optimal smooth' signal associated with  $x$  is given by:

$$y(B, x) := \arg \min_y \left\{ \|x - y\|_{H_1}^2 + \langle Ay, BAy \rangle_{H_2} \right\}, \quad (7)$$

where  $B : H_2 \rightarrow H_2$  is a smoothing operator, provided that

$$\langle Ah, BAh \rangle_{H_2} \geq 0, \quad h \in H_1.$$

### Definition

The optimal smoothing operator associated with the Hodrick-Prescott filter (6) is the minimizer of the difference between the optimal solution  $y(B, x)$ , and the conditional expectation  $E[y|x]$ , the best predictor of any signal  $y$  given the functional data  $x$ :

$$\hat{B} = \arg \min_B \|E[y|x] - y(B, x)\|_{H_1}^2. \quad (8)$$

## Proposition

Let  $A : H_1 \rightarrow H_2$  be a compact operator with the singular system  $(\lambda_n, e_n, d_n)$ . Assume further that the smoothing operator  $B : H_2 \rightarrow H_2$  is linear, bounded and satisfies

$$\langle Ah, BAh \rangle_{H_2} \geq 0, \quad h \in H_1. \quad (9)$$

Then, there exists a unique  $y(B, x) \in H_1$  which minimizes the functional

$$J_B(y) = \|x - y\|_{H_1}^2 + \langle Ay, BAy \rangle_{H_2}.$$

This minimizer is given by the formula

$$y(B, x) = (I_{H_1} + A^*BA)^{-1}x. \quad (10)$$

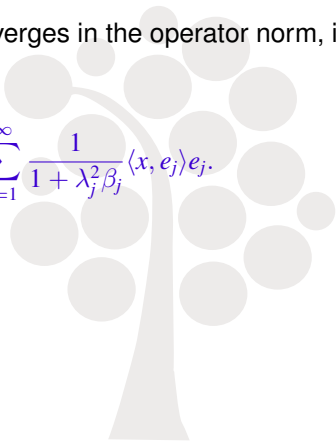


If the smoothing operator  $B : H_2 \rightarrow H_2$  admits the following representation

$$Bh = \sum_{k=1}^{\infty} \beta_k \langle h, d_k \rangle d_k, \quad h \in H_2, \quad (11)$$

where  $\beta_k > 0$ ,  $k = 1, 2, \dots$ , and the sum converges in the operator norm, i.e.  $B$  is linear, compact and injective, then

$$y(B, x) = (I_{H_1} + A^*BA)^{-1}x = \sum_{j=1}^{\infty} \frac{1}{1 + \lambda_j^2 \beta_j} \langle x, e_j \rangle e_j. \quad (12)$$



## Assumptions

- 1  $u$  and  $v$  are independent random variables with zero mean and covariance operators  $\Sigma_u$  and  $\Sigma_v$  respectively.
- 2 The independent random variables  $u$  and  $v$  are respectively  $N(0, \Sigma_u)$  and  $N(0, \Sigma_v)$  distributed, where the covariance operators  $\Sigma_u$  and  $\Sigma_v$  are positive-definite and trace class operators on  $H_1$  and  $H_2$  respectively.
- 3 The orthogonal (in  $H_1$ ) random variables  $\Pi u$  and  $(I_{H_1} - \Pi)u$  are independent:

$$\Pi \Sigma_u = \Sigma_u \Pi. \quad (13)$$

- 4 The operator

$$Q_v := A^*(AA^*)^{-1}\Sigma_v(AA^*)^{-1}A$$

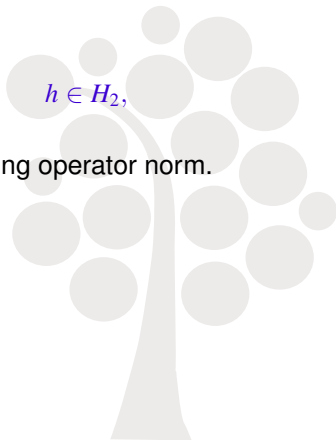
is trace class.

Since the covariance operators  $\Sigma_u$  and  $\Sigma_v$  are trace class and thus compact, by Riesz' Representation Theorem, they admit the following decompositions:

$$\Sigma_u h = \sum_{k=1}^{\infty} \mu_k \langle h, e_k \rangle e_k, \quad h \in H_1, \quad (14)$$

$$\Sigma_v h = \sum_{k=1}^{\infty} \tau_k \langle h, d_k \rangle d_k, \quad h \in H_2, \quad (15)$$

where the sums converge in the corresponding operator norm.



## Proposition

Let  $X, Y$  be jointly Gaussian  $H$ -valued random variables. Assume that both  $X$  and  $Y$  have means  $\mu_X$  and  $\mu_Y$ , and that the covariance of  $X$ ,  $\Sigma_X$ , is injective. Then, the conditional expectation of  $Y$  given  $X$  is

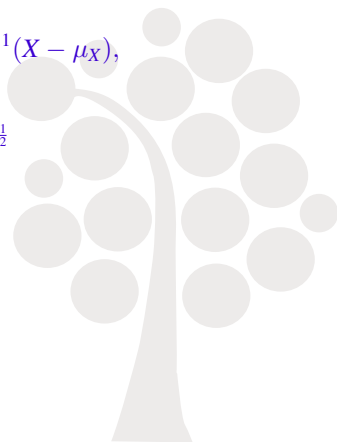
$$E[Y|X] = \mu_Y + \Sigma_{XY}\Sigma_X^{-1}(X - \mu_X), \quad (16)$$

provided that the operator

$$T = \Sigma_{XY}\Sigma_X^{-\frac{1}{2}} \quad (17)$$

is Hilbert-Schmidt.

See Mandelbaum [8]



## Theorem

Let Assumptions (1) to (4) hold, and that

$$\|T\|_2^2 = \sum_{k=1}^{\infty} \frac{\tau_k}{\lambda_k^2} \left( \frac{\lambda_k^2 \mu_k}{\tau_k} + 1 \right)^{-1} < \infty, \quad (18)$$

then, for all  $x \in H_1$ , the smoothing operator (which is linear, compact and injective)

$$\hat{B}h := (AA^*)^{-1}A\Sigma_u A^* \Sigma_v^{-1}h = \sum_{k=1}^{\infty} \frac{\mu_k}{\tau_k} \langle h, d_k \rangle d_k, \quad h \in H_2, \quad (19)$$

where, the sum converges in the operator norm, is the unique operator which satisfies

$$\hat{B} = \arg \min_B \|y(B, x) - E[y|x]\|_{H_1},$$

where the minimum is taken with respect to all linear bounded operators which satisfy the positivity condition (9).

Furthermore, we have

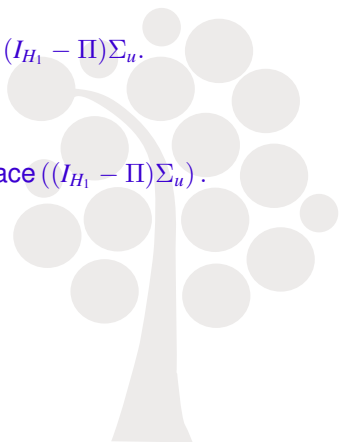
$$y(\hat{B}, x) - E[y|x] = (I_{H_1} - \Pi)(x - E[x]), \quad (20)$$

and its covariance operator is

$$\text{cov} (y(\hat{B}, x) - E[y|x]) = (I_{H_1} - \Pi)\Sigma_u. \quad (21)$$

In particular,

$$E \left( \|y(\hat{B}, x) - E[y|x]\|_{H_1}^2 \right) = \text{trace} ((I_{H_1} - \Pi)\Sigma_u). \quad (22)$$



Assume that  $u \sim N(0, \Sigma_u)$  and  $v \sim N(0, \Sigma_v)$  where  $\Sigma_u$  and  $\Sigma_v$  are self-adjoint positive-definite and bounded but not trace class operators on  $H_1$  and  $H_2$ , respectively.

Following Rozanov (1968), we can look at these Gaussian variables as generalized random variables on an appropriate Hilbert scale, where the covariance operators can be maximally extended to self-adjoint positive-definite, bounded and trace class operators on a larger space.

We first construct the Hilbert scales  $(H_1^n)_{n \in \mathbb{R}}$  ( $(H_2^n)_{n \in \mathbb{R}}$ ) induced by  $K_1 = (A^*A)^{-1}$  ( $K_2 = (AA^*)^{-1}$ ) of  $H_1$ , ( $H_2$ ).

For all  $n \in \mathbb{N}$  the space  $H_1^n$  is a complete space with respect to the norm induced by the following inner product

$$\langle x, y \rangle_{H_1^n} := \langle (A^*A)^{-n}x, (A^*A)^{-n}y \rangle_{H_1}, \quad x, y \in H_1^n. \quad (23)$$

**Assumption (5):** There is  $n_0 > 0$  such that for all  $n \geq n_0$  we have

$$\sum_{k=1}^{\infty} \lambda_k^{4n-2} \mu_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_k^{4n} \tau_k < \infty.$$

Under Assumption 5, the covariance operators  $\tilde{\Sigma}_u$ ,  $\tilde{\Sigma}$  and  $\tilde{\Sigma}_v$  are trace class on the Hilbert spaces  $H_1^{-n}$  and  $H_2^{-n}$ , respectively, where

$$\tilde{\Sigma}_u = (A^*A)^n \Sigma_u (A^*A)^n, \quad \tilde{\Sigma}_v = (AA^*)^n \Sigma_v (AA^*)^n \quad (24)$$

and

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_u + \tilde{Q}_v & \tilde{Q}_v \\ \tilde{Q}_v & \tilde{Q}_v \end{pmatrix}, \quad (25)$$

where,

$$\tilde{Q}_v := (A^*A)^n Q_v (A^*A)^n = A^* (AA^*)^{-1} \tilde{\Sigma}_v (AA^*)^{-1} A. \quad (26)$$



## Theorem

Let Assumption 5 hold. Then, the operator

$$\hat{B}h := (AA^*)^{-1}A\tilde{\Sigma}_uA^*\tilde{\Sigma}_v^{-1}h, \quad h \in H_2^{-n}, \quad (27)$$

is the unique optimal smoothing operator associated with the HP filter associated with  $H_1^{-n}$ -valued data  $x$ .



In this section we apply Theorem 7 to the case where  $u$  and  $v$  are white noise i.e.  $u$  and  $v$  are independent Gaussian random variables with zero means and covariance operators

$$\Sigma_u = \sigma_u I_{H_1}, \quad \Sigma_v = \sigma_v I_{H_2},$$

where  $I_{H_1}$  and  $I_{H_2}$  denotes the  $H_1$  and  $H_2$  identity operators, respectively and  $\sigma_u$  and  $\sigma_v$  are constant scalars. Assumption 5, reduces to

**Assumption 6.** There is an  $n_0 > 0$  such that  $\sum_{k=1}^{\infty} \lambda_k^{2(2n-1)} < \infty$  for all  $n \geq n_0$ . Under this assumption, the associated covariance operators  $\tilde{\Sigma}_u$ ,  $\tilde{\Sigma}_v$  and  $\tilde{Q}_v$  are all trace class operators. Hence, the expression (27) giving the optimal smoothing operator  $\hat{B}$  reduces to

$$\hat{B} = (AA^*)^{-1} A \Sigma_u A^* \Sigma_v^{-1} = \frac{\sigma_u}{\sigma_v} \sum_{k=1}^{\infty} \langle \cdot, d_k \rangle d_k = \frac{\sigma_u}{\sigma_v} I_{H_2^{-n}}, \quad (28)$$

i.e.  $\hat{B}$  is the noise-to-signal ratio.

## Assumptions

- 7 the linear operator  $A : H_1 \rightarrow H_2$  is
  - (1a) Closed and defined on a dense subspace  $\mathcal{D}(A)$  of  $H_1$ ,
  - (1b) Its range,  $\text{Ran}(A)$ , is closed.
- 8  $u$  and  $v$  are independent Gaussian random variables with zero mean and covariance operators  $\Sigma_u$  and  $\Sigma_v$  respectively.
- 9 The orthogonal (in  $H_1$ ) random variables  $\Pi u$  and  $(I_{H_1} - \Pi)u$  are independent:

$$\Pi \Sigma_u = \Sigma_u \Pi. \quad (29)$$

Assumption (1) is equivalent to the fact that  $A^\dagger$  is bounded.

## Proposition

Let  $A : H_1 \rightarrow H_2$  be a closed, linear operator and its domain is dense in  $H_1$ . Assume further the smoothing operator  $B : H_2 \rightarrow H_2$  is closed, densely defined and satisfies

$$\langle Ah, BAh \rangle_{H_2} \geq 0, \quad h \in H_1. \quad (30)$$

Then, there exists a unique  $y(B, x) \in H_1$  which minimizes the functional

$$J_B(y) = \|x - y\|_{H_1}^2 + \langle Ay, BAy \rangle_{H_2}.$$

This minimizer is given by the formula

$$y(B, x) = (I_{H_1} + A^*BA)^{-1}x. \quad (31)$$

Given Assumption (8), it holds that  $(x, y)$  is Gaussian with covariance operator

$$\Sigma = \begin{pmatrix} \Sigma_u + Q_v & Q_v \\ Q_v & Q_v \end{pmatrix}, \quad (32)$$

where,

$$Q_v := A^\dagger \Sigma_v (A^\dagger)^*. \quad (33)$$

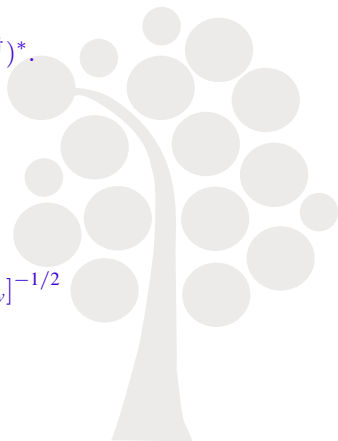
### Lemma

*The linear operator  $Q_v$  is trace class.*

*Moreover, the linear operator*

$$T := Q_v [\Sigma_u + Q_v]^{-1/2}$$

*is Hilbert-Schmidt.*



## Theorem

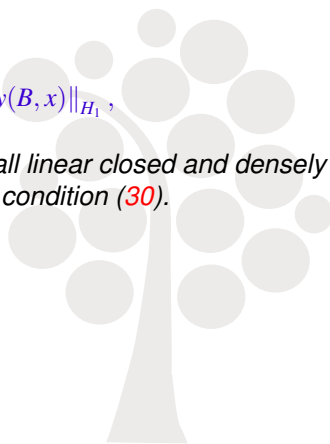
Under Assumptions (7), (8) and (9), the smoothing operator

$$\hat{B} := (A^\dagger)^* \Sigma_u A^* \Sigma_v^{-1} \quad (34)$$

is the unique operator which satisfies

$$\hat{B} = \arg \min_B \|E[y|x] - y(B, x)\|_{H_1},$$

where the minimum is taken with respect to all linear closed and densely defined operators which satisfy the positivity condition (30).



Assuming that  $u \sim N(0, \Sigma_u)$  and  $v \sim N(0, \Sigma_v)$  where  $\Sigma_u$  and  $\Sigma_v$  are self-adjoint positive-definite bounded but not trace class operators on  $H_1$  and  $H_2$ , respectively. In view of Assumption (7), the operator  $A^\dagger : H_2 \rightarrow H_1$  is linear and bounded operator. Put  $H_3 := \text{Ran}A$ ,  $H_3$  is a Hilbert space, since it is a closed subspace of Hilbert space  $H_2$ . Let  $\bar{A}^\dagger$  be the restriction of  $A^\dagger$  on  $H_3$  i.e.  $\bar{A}^\dagger : H_3 \rightarrow H_1$ . Hence  $\bar{A}^\dagger$  is injective bounded linear operator.

### Remark

In view of Hodrick-Prescott Filter (6),  $v \in \text{Ran}(A) = H_3$  i.e. it can be seen as  $H_3$ -random variable with covariance operator  $\Sigma_v : H_3 \rightarrow H_3$ .

Set

$$K_1 := (\bar{A}^\dagger (\bar{A}^\dagger)^*)^{-1} : H_1 \rightarrow H_1,$$

and

$$K_2 := ((\bar{A}^\dagger)^* \bar{A}^\dagger)^{-1} : H_3 \rightarrow H_3.$$

**Assumption (10):** There is  $n_0 > 0$  such that the covariance operators  $\tilde{\Sigma}_u, \tilde{\Sigma}$  and  $\tilde{\Sigma}_v$  are trace class on the Hilbert spaces  $H_1^{-n}$  and  $H_3^{-n}$ , respectively, where

$$\tilde{\Sigma}_u = (\bar{A}^\dagger (\bar{A}^\dagger)^*)^n \Sigma_u (\bar{A}^\dagger (\bar{A}^\dagger)^*)^n, \quad \tilde{\Sigma}_v = ((\bar{A}^\dagger)^* \bar{A}^\dagger)^n \Sigma_v ((\bar{A}^\dagger)^* \bar{A}^\dagger)^n \quad (35)$$

and

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_u + \tilde{Q}_v & \tilde{Q}_v \\ \tilde{Q}_v & \tilde{Q}_v \end{pmatrix}, \quad (36)$$

where,

$$\tilde{Q}_v := \bar{A}^\dagger \tilde{\Sigma}_v (\bar{A}^\dagger)^*. \quad (37)$$

### Theorem

Let assumption 5 hold. Then, the unique optimal smoothing operator associated with the HP filter associated with  $H_1^{-n}$ -valued data  $x$  is given by:

$$\hat{B}h := (\bar{A}^\dagger)^* \tilde{\Sigma}_u \bar{A}^* \tilde{\Sigma}_v^{-1} h, \quad h \in H_3^{-n}. \quad (38)$$



Assuming  $u$  and  $v$  independent and Gaussian random variables with zero means and covariance operators  $\Sigma_u = \sigma_u I_{H_1}$  and  $\Sigma_v = \sigma_v I_{H_3}$ , where  $I_{H_1}$  and  $I_{H_3}$  denote the  $H_1$  and  $H_3$  identity operators, respectively and  $\sigma_u$  and  $\sigma_v$  are constant scalars. Assumption 5 reduces to

**Assumption 11.** There is an  $n_0 > 0$  such that  $(\bar{A}^\dagger (\bar{A}^\dagger)^*)^{2n}$  and  $((\bar{A}^\dagger)^* \bar{A}^\dagger)^{2n}$  are trace class for all  $n \geq n_0$ .



Under this assumption, the associated covariance operators

$$\begin{aligned}\tilde{\Sigma}_u &= (\bar{A}^\dagger (\bar{A}^\dagger)^*)^n \Sigma_u (\bar{A}^\dagger (\bar{A}^\dagger)^*)^n = \sigma_u (\bar{A}^\dagger (\bar{A}^\dagger)^*)^{2n}, \\ \tilde{\Sigma}_v &= ((\bar{A}^\dagger)^* \bar{A}^\dagger)^n \Sigma_v ((\bar{A}^\dagger)^* \bar{A}^\dagger)^n = \sigma_v ((\bar{A}^\dagger)^* \bar{A}^\dagger)^{2n}\end{aligned}$$







and

$$\tilde{Q}_v = \sigma_v A^\dagger ((A^\dagger)^* A^\dagger)^{2n} (A^\dagger)^* = \sigma_v (A^\dagger (A^\dagger)^*)^{2n+1}$$

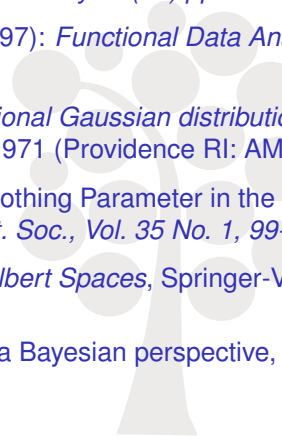







are trace class, the expression (38) giving the optimal smoothing operator  $\hat{B}$  reduces to

$$\hat{B} = (\bar{A}^\dagger)^* \tilde{\Sigma}_u A^* \tilde{\Sigma}_v^{-1} h = \frac{\sigma_u}{\sigma_v} I_{H_3^{-n}}, \quad (39)$$

i.e.  $\hat{B}$  is the noise-to-signal ratio which is in the same pattern as in the classical HP filter.

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**Thanks for your attention!**

