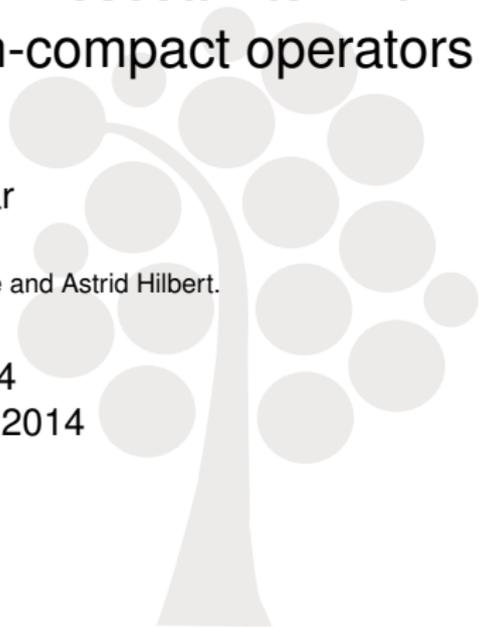


On the functional Hodrick-Prescott filter with compact operators and non-compact operators

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The univariate HP filter extracts a 'signal' $y(\alpha, x) = (y_1(\alpha, x), \dots, y_T(\alpha, x))$ from a noisy time series $x = (x_1, \dots, x_T)$ as a minimizer of

$$\sum_{t=1}^T (x_t - y_t)^2 + \alpha \sum_{t=3}^T (y_t - 2y_{t-1} + y_{t-2})^2, \quad (1)$$

with respect to $y = (y_1, \dots, y_T)$, for an appropriately chosen positive parameter α , called the smoothing parameter.

The second order differencing operator $Py(t) = y_t - 2y_{t-1} + y_{t-2}$ is written in vector form as the following $(T-2) \times T$ -matrix

$$P := \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & -2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

To determine an appropriate value of the smoothing parameter α , Hodrick and Prescott (1997) suggest the time series (x, y) satisfies the following linear mixed model:

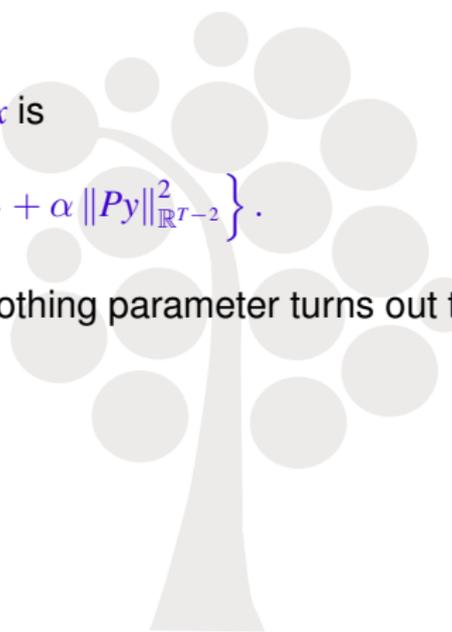
$$\begin{cases} x = y + u, \\ Py = v. \end{cases} \quad (2)$$

where, $u \sim N(0, \sigma_u^2 I_T)$ and $v \sim N(0, \sigma_v^2 I_{T-2})$.

The 'optimal smooth' signal associated with x is

$$\bar{y}(\alpha, x) := \arg \min_y \left\{ \|x - y\|_{\mathbb{R}^T}^2 + \alpha \|Py\|_{\mathbb{R}^{T-2}}^2 \right\}. \quad (3)$$

Using the model above, the appropriate smoothing parameter turns out to be the noise-to-signal ratio $\alpha^* = \sigma_u^2 / \sigma_v^2$.



Schlicht in (2005) proved that the noise-to-signal ratio satisfies

$$E[y|x] = y\left(\frac{\sigma_u^2}{\sigma_v^2}, x\right), \quad (4)$$

where $E[y|x]$ is the best predictor of any signal y given the time series x . Dermoune *et al.* proposed in (2009) an optimality criterion for choosing the smoothing parameter for the HP-filter. The smoothing parameter α is chosen as the following:

$$\alpha^* = \arg \min_{\alpha} \{ \|E[y|x] - y(\alpha, x)\|^2 \} \quad (5)$$

Furthermore, Dermoune *et al.* (2009) proposed a multivariate version of the HP filter and determined the possible optimal smoothing parameters.

Definition

Let H_1 and H_2 be two separable Hilbert spaces, with norms $\|\cdot\|_{H_i}$ and inner products $\langle \cdot, \cdot \rangle_{H_i}$, $i = 1, 2$, and $x \in H_1$ be a functional time series of observables. A functional Hodrick-Prescott filter reconstructs an 'optimal smooth signal' $y \in H_1$ that solves an equation $Ay = v$, corrupted by a noise v which is a priori unobservable, from observations x corrupted by a noise u which is also a priori unobservable:

$$\begin{cases} x = y + u, \\ Ay = v, \end{cases} \quad (6)$$

given the linear operator $A : H_1 \rightarrow H_2$ and u, v are independent random variables with zero mean and covariance operators Σ_u and Σ_v respectively.

The 'optimal smooth' signal associated with x is given by:

$$y(B, x) := \arg \min_y \left\{ \|x - y\|_{H_1}^2 + \langle Ay, BAy \rangle_{H_2} \right\}, \quad (7)$$

where $B : H_2 \rightarrow H_2$ is a smoothing operator, provided that

$$\langle Ah, BAh \rangle_{H_2} \geq 0, \quad h \in H_1.$$

Definition

The optimal smoothing operator associated with the Hodrick-Prescott filter (6) is the minimizer of the difference between the optimal solution $y(B, x)$, and the conditional expectation $E[y|x]$, the best predictor of any signal y given the functional data x :

$$\hat{B} = \arg \min_B \|E[y|x] - y(B, x)\|_{H_1}^2. \quad (8)$$

Proposition

Let $A : H_1 \rightarrow H_2$ be a compact operator with the singular system (λ_n, e_n, d_n) . Assume further that the smoothing operator $B : H_2 \rightarrow H_2$ is linear, bounded and satisfies

$$\langle Ah, BAh \rangle_{H_2} \geq 0, \quad h \in H_1. \quad (9)$$

Then, there exists a unique $y(B, x) \in H_1$ which minimizes the functional

$$J_B(y) = \|x - y\|_{H_1}^2 + \langle Ay, BAy \rangle_{H_2}.$$

This minimizer is given by the formula

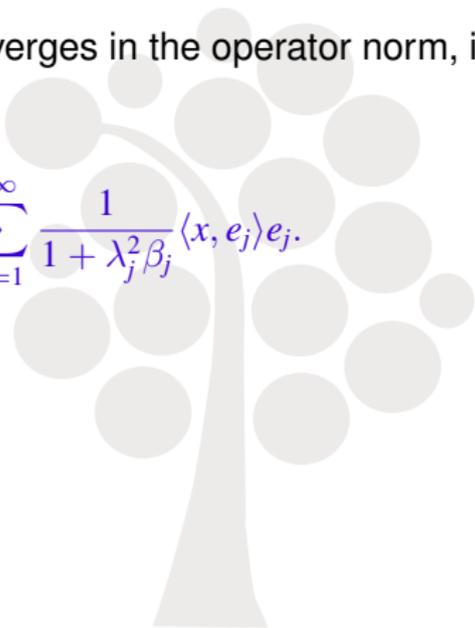
$$y(B, x) = (I_{H_1} + A^*BA)^{-1}x. \quad (10)$$

If the smoothing operator $B : H_2 \rightarrow H_2$ admits the following representation

$$Bh = \sum_{k=1}^{\infty} \beta_k \langle h, d_k \rangle d_k, \quad h \in H_2, \quad (11)$$

where $\beta_k > 0$, $k = 1, 2, \dots$, and the sum converges in the operator norm, i.e. B is linear, compact and injective, then

$$y(B, x) = (I_{H_1} + A^*BA)^{-1}x = \sum_{j=1}^{\infty} \frac{1}{1 + \lambda_j^2 \beta_j} \langle x, e_j \rangle e_j. \quad (12)$$



Assumptions

- 1 u and v are independent random variables with zero mean and covariance operators Σ_u and Σ_v respectively.
- 2 The independent random variables u and v are respectively $N(0, \Sigma_u)$ and $N(0, \Sigma_v)$ distributed, where the covariance operators Σ_u and Σ_v are positive-definite and trace class operators on H_1 and H_2 respectively.
- 3 The orthogonal (in H_1) random variables Πu and $(I_{H_1} - \Pi)u$ are independent:

$$\Pi \Sigma_u = \Sigma_u \Pi. \quad (13)$$

- 4 The operator

$$Q_v := A^*(AA^*)^{-1}\Sigma_v(AA^*)^{-1}A$$

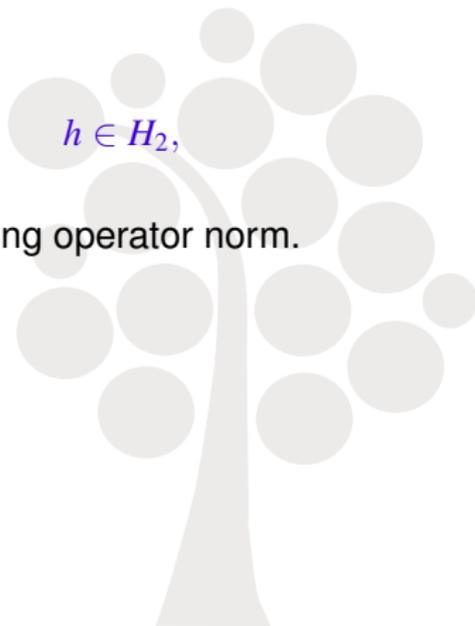
is trace class.

Since the covariance operators Σ_u and Σ_v are trace class and thus compact, by Riesz' Representation Theorem, they admit the following decompositions:

$$\Sigma_u h = \sum_{k=1}^{\infty} \mu_k \langle h, e_k \rangle e_k, \quad h \in H_1, \quad (14)$$

$$\Sigma_v h = \sum_{k=1}^{\infty} \tau_k \langle h, d_k \rangle d_k, \quad h \in H_2, \quad (15)$$

where the sums converge in the corresponding operator norm.



Proposition

Let X, Y be jointly Gaussian H -valued random variables. Assume that both X and Y have means μ_X and μ_Y , and that the covariance of X , Σ_X , is injective. Then, the conditional expectation of Y given X is

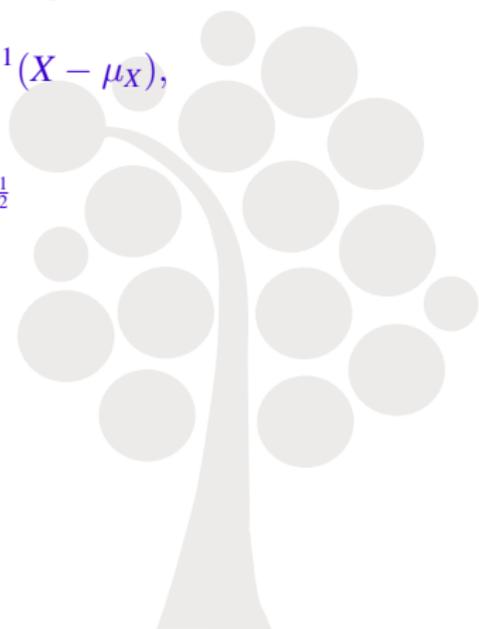
$$E[Y|X] = \mu_Y + \Sigma_{XY}\Sigma_X^{-1}(X - \mu_X), \quad (16)$$

provided that the operator

$$T = \Sigma_{XY}\Sigma_X^{-\frac{1}{2}} \quad (17)$$

is Hilbert-Schmidt.

See Mandelbaum [8]



Theorem

Let Assumptions (1) to (4) hold, and that

$$\|T\|_2^2 = \sum_{k=1}^{\infty} \frac{\tau_k}{\lambda_k^2} \left(\frac{\lambda_k^2 \mu_k}{\tau_k} + 1 \right)^{-1} < \infty, \quad (18)$$

then, for all $x \in H_1$, the smoothing operator (which is linear, compact and injective)

$$\hat{B}h := (AA^*)^{-1}A\Sigma_u A^* \Sigma_v^{-1}h = \sum_{k=1}^{\infty} \frac{\mu_k}{\tau_k} \langle h, d_k \rangle d_k, \quad h \in H_2, \quad (19)$$

where, the sum converges in the operator norm, is the unique operator which satisfies

$$\hat{B} = \arg \min_B \|y(B, x) - E[y|x]\|_{H_1},$$

where the minimum is taken with respect to all linear bounded operators which satisfy the positivity condition (9).

Furthermore, we have

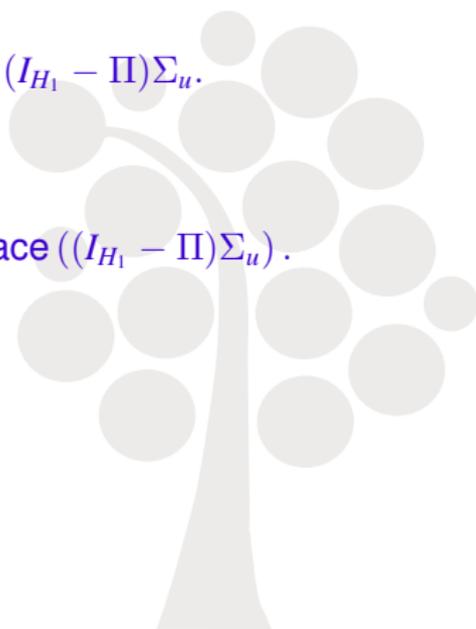
$$y(\hat{B}, x) - E[y|x] = (I_{H_1} - \Pi)(x - E[x]), \quad (20)$$

and its covariance operator is

$$\text{cov} (y(\hat{B}, x) - E[y|x]) = (I_{H_1} - \Pi)\Sigma_u. \quad (21)$$

In particular,

$$E \left(\|y(\hat{B}, x) - E[y|x]\|_{H_1}^2 \right) = \text{trace} ((I_{H_1} - \Pi)\Sigma_u). \quad (22)$$



Assume that $u \sim N(0, \Sigma_u)$ and $v \sim N(0, \Sigma_v)$ where Σ_u and Σ_v are self-adjoint positive-definite and bounded but not trace class operators on H_1 and H_2 , respectively.

Following Rozanov (1968), we can look at these Gaussian variables as generalized random variables on an appropriate Hilbert scale, where the covariance operators can be maximally extended to self-adjoint positive-definite, bounded and trace class operators on a larger space.

We first construct the Hilbert scales $(H_1^n)_{n \in \mathbb{R}}$ ($(H_2^n)_{n \in \mathbb{R}}$) induced by $K_1 = (A^*A)^{-1}$ ($K_2 = (AA^*)^{-1}$) of H_1 , (H_2).

For all $n \in \mathbb{N}$ the space H_1^n is a complete space with respect to the norm induced by the following inner product

$$\langle x, y \rangle_{H_1^n} := \langle (A^*A)^{-n}x, (A^*A)^{-n}y \rangle_{H_1}, \quad x, y \in H_1^n. \quad (23)$$

Assumption (5): There is $n_0 > 0$ such that for all $n \geq n_0$ we have

$$\sum_{k=1}^{\infty} \lambda_k^{4n-2} \mu_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_k^{4n} \tau_k < \infty.$$

Under Assumption 5, the covariance operators $\tilde{\Sigma}_u$, $\tilde{\Sigma}$ and $\tilde{\Sigma}_v$ are trace class on the Hilbert spaces H_1^{-n} and H_2^{-n} , respectively, where

$$\tilde{\Sigma}_u = (A^*A)^n \Sigma_u (A^*A)^n, \quad \tilde{\Sigma}_v = (AA^*)^n \Sigma_v (AA^*)^n \quad (24)$$

and

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_u + \tilde{Q}_v & \tilde{Q}_v \\ \tilde{Q}_v & \tilde{Q}_v \end{pmatrix}, \quad (25)$$

where,

$$\tilde{Q}_v := (A^*A)^n Q_v (A^*A)^n = A^* (AA^*)^{-1} \tilde{\Sigma}_v (AA^*)^{-1} A. \quad (26)$$

Theorem

Let Assumption 5 hold. Then, the operator

$$\hat{B}h := (AA^*)^{-1}A\tilde{\Sigma}_uA^*\tilde{\Sigma}_v^{-1}h, \quad h \in H_2^{-n}, \quad (27)$$

is the unique optimal smoothing operator associated with the HP filter associated with H_1^{-n} -valued data x .



In this section we apply Theorem 7 to the case where u and v are white noise i.e. u and v are independent Gaussian random variables with zero means and covariance operators

$$\Sigma_u = \sigma_u I_{H_1}, \quad \Sigma_v = \sigma_v I_{H_2},$$

where I_{H_1} and I_{H_2} denotes the H_1 and H_2 identity operators, respectively and σ_u and σ_v are constant scalars. Assumption 5, reduces to

Assumption 6. There is an $n_0 > 0$ such that $\sum_{k=1}^{\infty} \lambda_k^{2(2n-1)} < \infty$ for all $n \geq n_0$. Under this assumption, the associated covariance operators $\tilde{\Sigma}_u$, $\tilde{\Sigma}_v$ and \tilde{Q}_v are all trace class operators. Hence, the expression (27) giving the optimal smoothing operator \hat{B} reduces to

$$\hat{B} = (AA^*)^{-1} A \Sigma_u A^* \Sigma_v^{-1} = \frac{\sigma_u}{\sigma_v} \sum_{k=1}^{\infty} \langle \cdot, d_k \rangle d_k = \frac{\sigma_u}{\sigma_v} I_{H_2^{-n}}, \quad (28)$$

i.e. \hat{B} is the noise-to-signal ratio.

Assumptions

- 7 the linear operator $A : H_1 \rightarrow H_2$ is
 - (1a) Closed and defined on a dense subspace $\mathcal{D}(A)$ of H_1 ,
 - (1b) Its range, $\text{Ran}(A)$, is closed.
- 8 u and v are independent Gaussian random variables with zero mean and covariance operators Σ_u and Σ_v respectively.
- 9 The orthogonal (in H_1) random variables Πu and $(I_{H_1} - \Pi)u$ are independent:

$$\Pi \Sigma_u = \Sigma_u \Pi. \quad (29)$$

Assumption (1) is equivalent to the fact that A^\dagger is bounded.

Proposition

Let $A : H_1 \rightarrow H_2$ be a closed, linear operator and its domain is dense in H_1 . Assume further the smoothing operator $B : H_2 \rightarrow H_2$ is closed, densely defined and satisfies

$$\langle Ah, BAh \rangle_{H_2} \geq 0, \quad h \in H_1. \quad (30)$$

Then, there exists a unique $y(B, x) \in H_1$ which minimizes the functional

$$J_B(y) = \|x - y\|_{H_1}^2 + \langle Ay, BAy \rangle_{H_2}.$$

This minimizer is given by the formula

$$y(B, x) = (I_{H_1} + A^*BA)^{-1}x. \quad (31)$$

Given Assumption (8), it holds that (x, y) is Gaussian with covariance operator

$$\Sigma = \begin{pmatrix} \Sigma_u + Q_v & Q_v \\ Q_v & Q_v \end{pmatrix}, \quad (32)$$

where,

$$Q_v := A^\dagger \Sigma_v (A^\dagger)^*. \quad (33)$$

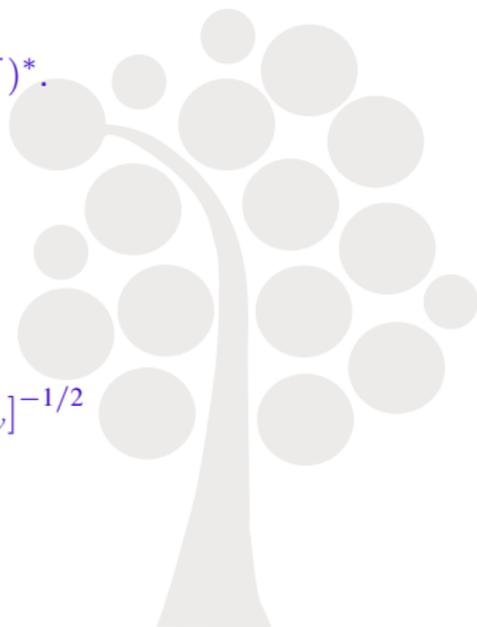
Lemma

The linear operator Q_v is trace class.

Moreover, the linear operator

$$T := Q_v [\Sigma_u + Q_v]^{-1/2}$$

is Hilbert-Schmidt.



Theorem

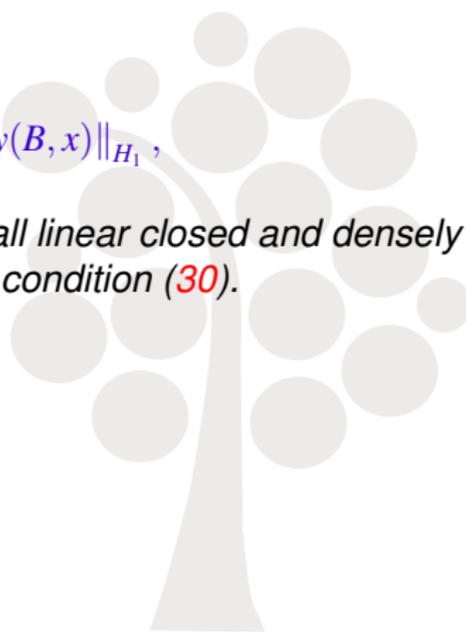
Under Assumptions (7), (8) and (9), the smoothing operator

$$\hat{B} := (A^\dagger)^* \Sigma_u A^* \Sigma_v^{-1} \quad (34)$$

is the unique operator which satisfies

$$\hat{B} = \arg \min_B \|E[y|x] - y(B, x)\|_{H_1},$$

where the minimum is taken with respect to all linear closed and densely defined operators which satisfy the positivity condition (30).



Assuming that $u \sim N(0, \Sigma_u)$ and $v \sim N(0, \Sigma_v)$ where Σ_u and Σ_v are self-adjoint positive-definite bounded but not trace class operators on H_1 and H_2 , respectively. In view of Assumption (7), the operator $A^\dagger : H_2 \rightarrow H_1$ is linear and bounded operator. Put $H_3 := \text{Ran}A$, H_3 is a Hilbert space, since it is a closed subspace of Hilbert space H_2 . Let \bar{A}^\dagger be the restriction of A^\dagger on H_3 i.e. $\bar{A}^\dagger : H_3 \rightarrow H_1$. Hence \bar{A}^\dagger is injective bounded linear operator.

Remark

In view of Hodrick-Prescott Filter (6), $v \in \text{Ran}(A) = H_3$ i.e. it can be seen as H_3 -random variable with covariance operator $\Sigma_v : H_3 \rightarrow H_3$.

Set

$$K_1 := (\bar{A}^\dagger (\bar{A}^\dagger)^*)^{-1} : H_1 \rightarrow H_1,$$

and

$$K_2 := ((\bar{A}^\dagger)^* \bar{A}^\dagger)^{-1} : H_3 \rightarrow H_3.$$

Assumption (10): There is $n_0 > 0$ such that the covariance operators $\tilde{\Sigma}_u, \tilde{\Sigma}$ and $\tilde{\Sigma}_v$ are trace class on the Hilbert spaces H_1^{-n} and H_3^{-n} , respectively, where

$$\tilde{\Sigma}_u = (\bar{A}^\dagger (\bar{A}^\dagger)^*)^n \Sigma_u (\bar{A}^\dagger (\bar{A}^\dagger)^*)^n, \quad \tilde{\Sigma}_v = ((\bar{A}^\dagger)^* \bar{A}^\dagger)^n \Sigma_v ((\bar{A}^\dagger)^* \bar{A}^\dagger)^n \quad (35)$$

and

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_u + \tilde{Q}_v & \tilde{Q}_v \\ \tilde{Q}_v & \tilde{Q}_v \end{pmatrix}, \quad (36)$$

where,

$$\tilde{Q}_v := \bar{A}^\dagger \tilde{\Sigma}_v (\bar{A}^\dagger)^*. \quad (37)$$

Theorem

Let assumption 5 hold. Then, the unique optimal smoothing operator associated with the HP filter associated with H_1^{-n} -valued data x is given by:

$$\hat{B}h := (\bar{A}^\dagger)^* \tilde{\Sigma}_u \bar{A}^* \tilde{\Sigma}_v^{-1} h, \quad h \in H_3^{-n}. \quad (38)$$

Assuming u and v independent and Gaussian random variables with zero means and covariance operators $\Sigma_u = \sigma_u I_{H_1}$ and $\Sigma_v = \sigma_v I_{H_3}$, where I_{H_1} and I_{H_3} denote the H_1 and H_3 identity operators, respectively and σ_u and σ_v are constant scalars. Assumption 5 reduces to

Assumption 11. There is an $n_0 > 0$ such that $(\bar{A}^\dagger (\bar{A}^\dagger)^*)^{2n}$ and $((\bar{A}^\dagger)^* \bar{A}^\dagger)^{2n}$ are trace class for all $n \geq n_0$.



Under this assumption, the associated covariance operators

$$\begin{aligned}\tilde{\Sigma}_u &= (\bar{A}^\dagger (\bar{A}^\dagger)^*)^n \Sigma_u (\bar{A}^\dagger (\bar{A}^\dagger)^*)^n = \sigma_u (\bar{A}^\dagger (\bar{A}^\dagger)^*)^{2n}, \\ \tilde{\Sigma}_v &= ((\bar{A}^\dagger)^* \bar{A}^\dagger)^n \Sigma_v ((\bar{A}^\dagger)^* \bar{A}^\dagger)^n = \sigma_v ((\bar{A}^\dagger)^* \bar{A}^\dagger)^{2n}\end{aligned}$$

and

$$\tilde{Q}_v = \sigma_v A^\dagger ((A^\dagger)^* A^\dagger)^{2n} (A^\dagger)^* = \sigma_v (A^\dagger (A^\dagger)^*)^{2n+1}$$

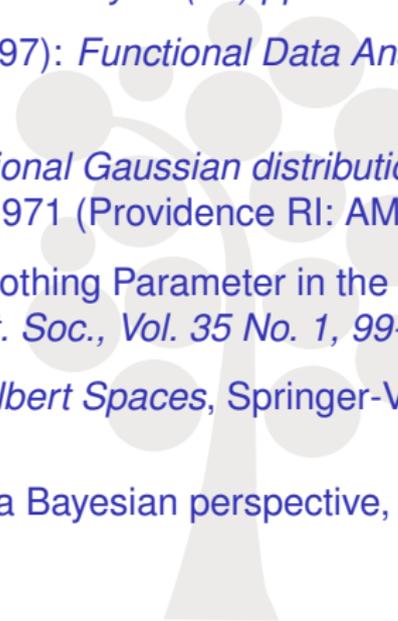
are trace class, the expression (38) giving the optimal smoothing operator \hat{B} reduces to

$$\hat{B} = (\bar{A}^\dagger)^* \tilde{\Sigma}_u A^* \tilde{\Sigma}_v^{-1} h = \frac{\sigma_u}{\sigma_v} I_{H_3^{-n}}, \quad (39)$$

i.e. \hat{B} is the noise-to-signal ratio which is in the same pattern as in the classical HP filter.

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Thanks for your attention!

