Formulas, equalities and inequalities for covariance matrices of estimations under a general linear model

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Outline

- Statistical motivations.
- How to characterize general matrix equalities/inequalities.
- Structures of covariance matrices of estimations (BLUEs).
- Formulas for ranks/inertias of BLUEs'/OLSEs' covariance matrices.
- Equalities/inequalities of BLUEs'/OLSEs' covariance matrices.
- Some applications under two transformed models.

Backgrounds

Consider a general linear regression model

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon, \quad E(\varepsilon) = \mathbf{0}, \quad Cov(\varepsilon) = \mathbf{\Sigma}, \tag{1}$$

where

$\mathbf{y} \in \mathbb{R}^{n\times 1}$ is an observable random vector,
$\mathbf{X} \in \mathbb{R}^{n\times p}$ is a known matrix of arbitrary rank,
$\beta \in \mathbb{R}^{p\times 1}$ is a vector of unknown parameters,
$\varepsilon \in \mathbb{R}^{n\times 1}$ is an unobservable random vector,
$\mathbf{\Sigma} \in \mathbb{R}^{n\times n}$ is a known or unknown nnd matrix.

Backgrounds

Assume that a linear estimation **Gy** is given under a general linear model. Then the covariance matrix of **Gy** is which is quadratic matrix function with respect to **G**. If **G** is not unique, e.g., it involves some variable matrix, then

$$Cov(\mathbf{GY}) = \mathbf{G\Sigma G}'$$

may vary with respect to the choice of **G**.

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may vary with respect to the choice of **G**.

Covariance matrices of estimations plays essential roles in determining properties of estimations. In particular, equalities and inequalities for covariance matrices can be utilized in the comparison of optimality of estimations under various assumptions. Hence, it is desirable to establish as many as possible equalities and inequalities of covariance matrices of estimations.

General forms of equalities and inequalities of covariance matrices of estimations $\mathbf{Gy}$ can simply be written as

$$Cov(\mathbf{Gy}) = \mathbf{A},$$
$$Cov(\mathbf{Gy}) \succ \mathbf{A}\, (\succcurlyeq \mathbf{A}, \prec \mathbf{A}, \preccurlyeq \mathbf{A}),$$

where $\mathbf{A}$ is any given symmetric matrix. If for example, $\mathbf{Gy}$ is taken as the best-known BLUE of an estimable $\mathbf{K}\beta$ under linear model, the above become

$$Cov[\,\mathrm{BLUE}(\mathbf{K}\beta)] = \mathbf{A},$$
$$Cov[\,\mathrm{BLUE}(\mathbf{K}\beta)] \succ \mathbf{A}\, (\succcurlyeq \mathbf{A}, \prec \mathbf{A}, \preccurlyeq \mathbf{A}).$$

Although the equalities and inequalities are easy to understand, $Cov[\mathrm{BLUE}(\mathbf{K}\beta)]$ in fact involves some complicated matrix operations of the given matrices and their generalized inverses.

From stat point of view, the above are a really work in stat of describing properties of estimations.

From math point of view, the above are a really work in math of solving matrix equations/matrix inequalities for a given matrix $\mathbf{A}$.

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A useful method for establishing matrix equalities/inequalities in matrix theory was sufficiently developed, which is based on the following lemma onranks/inertias of matrices.

Lemma

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, or $\mathbf{A} = \mathbf{A}' \in \mathbb{R}^{m\times m}$. Then by definition the following hold.

(a) $\mathbf{A} = \mathbf{0}$ *iff* $rank(\mathbf{A}) = 0$.

(b) $\mathbf{A}$ *is nonsingular iff* $rank(\mathbf{A}) = m$.

(c) $\mathbf{A} \succ \mathbf{0}$ *iff* $i_+(\mathbf{A}) = m$.

(d) $\mathbf{A} \prec \mathbf{0}$ *iff* $(i_-(\mathbf{A}) = m$.

(e) $\mathbf{A} \succcurlyeq \mathbf{0}$ *iff* $i_-(\mathbf{A}) = 0$.

(f) $\mathbf{A} \succcurlyeq \mathbf{0} \preccurlyeq \mathbf{0}$ *iff* $i_+(\mathbf{A}) = 0)$.

Lemma

Let $\mathbf{A},\ \mathbf{B} \in \mathbb{R}^{m\times n}$, $\mathbf{A} = \mathbf{A}',\ \mathbf{B} = \mathbf{B}' \in \mathbb{R}^{m\times m}$. *Then the following hold.*

(a) $\mathbf{A} = \mathbf{B}$ *iff* $r(\mathbf{A} - \mathbf{B}) = 0$.

(b) $\mathbf{A} - \mathbf{B}$ *is nonsingular iff* $r(\mathbf{A} - \mathbf{B}) = m$.

(c) $\mathbf{A} \succ \mathbf{B}$ $(\mathbf{A} \prec \mathbf{B})$ *iff* $i_{+}(\mathbf{A} - \mathbf{B}) = m$ $(i_{-}(\mathbf{A} - \mathbf{B}) = m)$.

(d) $\mathbf{A} \succcurlyeq \mathbf{B}$ $(\mathbf{A} \preccurlyeq \mathbf{B})$ *iff* $i_{-}(\mathbf{A} - \mathbf{B}) = 0$ $(i_{+}(\mathbf{A} - \mathbf{B}) = 0)$.

This lemma implies that

If some formulas are established for calculating ranks and inertias of covariance matrices of estimations under linear models, we can utilize them to derive equalities and inequalities of the covariance matrices, and to compare estimations under regression models.

Motivated by this lemma, a group of explicit formulas for calculating the rank and inertia for the difference

$$r\{\mathbf{A} - Cov[\,\mathrm{BLUE}(\mathbf{K}\beta)]\},$$
$$i_{\pm}\{\mathbf{A} - Cov[\,\mathrm{BLUE}(\mathbf{K}\beta)]\}$$

can luckily be established.

As consequences of these formulas, necessary and sufficient conditions can be established for the previous equalities and inequalities to hold, respectively.

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Various applications of the equalities/inequalities for the different choices of the two matrices $\mathbf{K}$ and $\mathbf{A}$.

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Various applications of the equalities/inequalities for the different choices of the two matrices $\mathbf{K}$ and $\mathbf{A}$.

Also recall that the residual vector with respect to $\mathrm{BLUE}(\mathbf{X}\beta)$ is

$$\mathbf{e} = \mathbf{y} - \mathrm{BLUE}(\mathbf{X}\beta),$$

formulas can be established for calculating

$$i_{\pm}[\mathbf{A} - Cov(\mathbf{e})],$$

and identifying conditions for

$$Cov(\mathbf{e}) = \mathbf{A} \ (\succ \mathbf{A} \ \succcurlyeq \mathbf{A}, \prec \mathbf{A}, \preccurlyeq \mathbf{A})$$

to hold can be derived.

The following are some known results on ranks/inertias of matrices (Marsaglia & Styan 1974).

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Lemma

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$. *Then*

$$r[\,\mathbf{A},\, \mathbf{B}\,] = r(\mathbf{A}) + r(\mathbf{E_A B}) = r(\mathbf{B}) + r(\mathbf{E_B A}),$$

$$r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = r(\mathbf{A}) + r(\mathbf{C F_A}) = r(\mathbf{C}) + r(\mathbf{A F_C}),$$

$$r\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = r(\mathbf{B}) + r(\mathbf{C}) + r(\mathbf{E_B A F_C}).$$

Lemma

Let $\mathbf{A} = \mathbf{A}' \in \mathbb{R}^{m\times m}$, $\mathbf{B} = \mathbf{B}' \in \mathbb{R}^{n\times n}$, $\mathbf{Q} \in \mathbb{R}^{m\times n}$, *and assume that* $\mathbf{P} \in \mathbb{R}^{m\times m}$ *is nonsingular. Then*

$$\begin{aligned}
& i_{\pm}(\mathbf{PAP}') = i_{\pm}(\mathbf{A}), \\
& i_{\pm}(\mathbf{A}^{+}) = i_{\pm}(\mathbf{A}), \quad i_{\pm}(-\mathbf{A}) = i_{\mp}(\mathbf{A}), \\
& i_{\pm}\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} = i_{\pm}(\mathbf{A}) + i_{\pm}(\mathbf{B}), \\
& i_{+}\begin{bmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{0} \end{bmatrix} = i_{-}\begin{bmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{0} \end{bmatrix} = r(\mathbf{Q}).
\end{aligned}$$

Lemma

(Tian 2010) Let $\mathbf{A} = \mathbf{A}' \in \mathbb{R}^{m\times m}$, $\mathbf{B} \in \mathbb{R}^{m\times n}$, *and* $\mathbf{D} = \mathbf{D}' \in \mathbb{R}^{n\times n}$. *Then*

$$i_{\pm}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{0} \end{bmatrix} = r(\mathbf{B}) + i_{\pm}(\mathbf{E_B A E_B}),$$

$$i_{\pm}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{D} \end{bmatrix} = i_{\pm}(\mathbf{A}) + i_{\pm}\begin{bmatrix} \mathbf{0} & \mathbf{E_A B} \\ \mathbf{B}'\mathbf{E_A} & \mathbf{D} - \mathbf{B}'\mathbf{A}^{+}\mathbf{B} \end{bmatrix}.$$

If $\mathbf{A} \succcurlyeq \mathbf{0}$, *then*

$$i_{+}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{0} \end{bmatrix} = r[\,\mathbf{A}, \,\mathbf{B}\,], \quad i_{-}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{0} \end{bmatrix} = r(\mathbf{B}).$$

Lemma

(Tian 2010) Let $\mathbf{A} = \mathbf{A}' \in \mathbb{R}^{m\times m}$, $\mathbf{B} \in \mathbb{R}^{q\times n}$, $\mathbf{D} = \mathbf{D}' \in \mathbb{R}^{n\times n}$ *and* $\mathbf{P} \in \mathbb{R}^{q\times m}$ *with* $\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{P}')$, *and* $\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{P})$. *Also let*

$$\mathbf{M} = \begin{bmatrix} -\mathbf{A} & \mathbf{P}' & \mathbf{0} \\ \mathbf{P} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{B}' & \mathbf{D} \end{bmatrix}.$$

Then

$$i_{\pm}[\,\mathbf{D} - \mathbf{B}'(\mathbf{P}')^{+}\mathbf{A}\mathbf{P}^{+}\mathbf{B}\,] = i_{\pm}(\mathbf{M}) - r(\mathbf{P}).$$

Lemma

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Then

$$i_{\pm}[\,\mathbf{D} - \mathbf{B}'(\mathbf{P}')^{+}\mathbf{A}\mathbf{P}^{+}\mathbf{B}\,] = i_{\pm}(\mathbf{M}) - r(\mathbf{P}).$$

Hence,

$$\mathbf{B}'(\mathbf{P}')^{+}\mathbf{A}\mathbf{P}^{+}\mathbf{B} \succcurlyeq \mathbf{D} \Leftrightarrow i_{+}(\mathbf{M}) = r(\mathbf{P}),$$
$$\mathbf{B}'(\mathbf{P}')^{+}\mathbf{A}\mathbf{P}^{+}\mathbf{B} \preccurlyeq \mathbf{D} \Leftrightarrow i_{-}(\mathbf{M}) = r(\mathbf{P}).$$
$$\mathbf{B}'(\mathbf{P}')^{+}\mathbf{A}\mathbf{P}^{+}\mathbf{B} = \mathbf{D} \Leftrightarrow r(\mathbf{M}) = 2r(\mathbf{P}),$$

The following result was originated from Drygas (1970) and Rao (1973).

Lemma

$$\mathbf{L}_0\mathbf{y} = \mathrm{BLUE}(\mathbf{K}\beta) \Leftrightarrow \mathbf{L}_0[\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}}\,] = [\,\mathbf{K},\, \mathbf{0}\,].$$

This equation is always consistent, that is,

$$[\,\mathbf{K},\, \mathbf{0}\,][\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}}\,]^{+}[\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}}\,] = [\,\mathbf{K},\, \mathbf{0}\,],$$

and its general solution, denoted by $\mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{\Sigma}}$, *can be written in the following parametric form*

$$\mathbf{L}_0 = \mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{\Sigma}} = [\,\mathbf{K},\, \mathbf{0}\,][\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}}\,]^{+} + \mathbf{U}(\,\mathbf{I}_n - [\,\mathbf{X},\, \mathbf{\Sigma}\,][\,\mathbf{X},\, \mathbf{\Sigma}\,]^{+}),$$

where $\mathbf{U} \in \mathbb{R}^{k\times n}$ *is arbitrary.*

Theorem

Let

$$\mathbf{M} = \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & -\mathbf{A} & \mathbf{K} \\ \mathbf{X}' & \mathbf{K}' & \mathbf{0} \end{bmatrix}.$$

Then the following three formulas hold

$$i_{+}\{\, \mathbf{A} - Cov[\,\mathrm{BLUE}(\mathbf{K}\beta)]\,\} = i_{-}(\mathbf{M}) - r(\mathbf{X}),$$
$$i_{-}\{\, \mathbf{A} - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\,\} = i_{+}(\mathbf{M}) - r[\,\mathbf{X}, \,\mathbf{\Sigma}\,],$$
$$r\{\, \mathbf{A} - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\,\} = r(\mathbf{M}) - r[\,\mathbf{X}, \,\mathbf{\Sigma}\,] - r(\mathbf{X}).$$

Theorem

In consequence, the following hold.

(a) $Cov[\mathrm{BLUE}(\mathbf{K}\beta)] \succ \mathbf{A} \Leftrightarrow i_{+}(\mathbf{M}) = r[\,\mathbf{X}, \, \mathbf{\Sigma}\,] + k$.

(b) $Cov[\mathrm{BLUE}(\mathbf{K}\beta)] \prec \mathbf{A} \Leftrightarrow i_{-}(\mathbf{M}) = r(\mathbf{X}) + k$.

(c) $Cov[\mathrm{BLUE}(\mathbf{K}\beta)] \succcurlyeq \mathbf{A} \Leftrightarrow i_{-}(\mathbf{M}) = r(\mathbf{X})$.

(d) $Cov[\mathrm{BLUE}(\mathbf{K}\beta)] \preccurlyeq \mathbf{A} \Leftrightarrow i_{+}(\mathbf{M}) = r[\,\mathbf{X}, \, \mathbf{\Sigma}\,]$.

(e) $Cov[\mathrm{BLUE}(\mathbf{K}\beta)] = \mathbf{A} \Leftrightarrow r(\mathbf{M}) = r[\,\mathbf{X}, \, \mathbf{\Sigma}\,] + r(\mathbf{X})$.

Proof.

Note that

$$Cov[\mathrm{BLUE}(\mathbf{K}\beta)] = [\,\mathbf{K},\, \mathbf{0}\,][\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}}\,]^{+}\mathbf{\Sigma}[\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}}\,]'^{+}[\,\mathbf{K},\, \mathbf{0}\,]'.$$

Hence,

$$\begin{aligned}&\mathbf{A} - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\\ &= \mathbf{A} - [\,\mathbf{K},\, \mathbf{0}\,][\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}}\,]^{+}\mathbf{\Sigma}([\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}}\,]')^{+}[\,\mathbf{K},\, \mathbf{0}\,]',\end{aligned}$$

and

$$\begin{aligned}&i_{\pm}\{\,\mathbf{A} - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\,\}\\ &= i_{\pm}\{\mathbf{A} - [\,\mathbf{K},\, \mathbf{0}\,][\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}}\,]^{+}\mathbf{\Sigma}([\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}}\,]')^{+}[\,\mathbf{K},\, \mathbf{0}\,]'\}.\end{aligned}$$

Applying previous formulas and simplifying will finally yields

$$i_{+}\{\, \mathbf{A} - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\ \} = i_{-}(\mathbf{M}) - r(\mathbf{X}),$$
$$i_{-}\{\, \mathbf{A} - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\,\} = i_{+}(\mathbf{M}) - r[\,\mathbf{X}, \, \boldsymbol{\Sigma}\,].$$

Adding the two equalities yields the third rank formulas. Setting the equalities equal to zero yields (a)–(e).

Special cases

Several special cases, as examples, are presented below for different choices of $\mathbf{K}$ and $\mathbf{A}$ in the formulas (home assignments!).

Theorem

Let

$$\mathbf{M} = \begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & -\mathbf{A} & \mathbf{X} \\ \mathbf{X}' & \mathbf{X}' & \mathbf{0} \end{bmatrix}.$$

Then the following inertia and rank formulas hold

$$i_{+}\{\, \mathbf{A} - Cov[\mathrm{BLUE}(\mathbf{X}\beta)] \,\} = i_{-}(\mathbf{M}) - r(\mathbf{X}),$$
$$i_{-}\{\, \mathbf{A} - Cov[\mathrm{BLUE}(\mathbf{X}\beta)] \,\} = i_{+}(\mathbf{M}) - r[\,\mathbf{X}, \,\boldsymbol{\Sigma}\,].$$

Theorem

If β is estimable, then

$$i_+\{\,\mathbf{A} - Cov[\mathrm{BLUE}(\beta)]\,\} = i_-(\,\boldsymbol{\Sigma} - \mathbf{XAX}'\,),$$
$$i_-\{\,\mathbf{A} - Cov[\mathrm{BLUE}(\beta)]\,\} = p + i_+(\,\boldsymbol{\Sigma} - \mathbf{XAX}'\,) - r[\,\mathbf{X},\,\boldsymbol{\Sigma}\,].$$

In consequence, the following hold.

(a) $Cov[\mathrm{BLUE}(\mathbf{X}\beta)] \succ \mathbf{A} \Leftrightarrow i_+(\mathbf{M}) = r[\,\mathbf{X},\,\boldsymbol{\Sigma}\,] + n.$

(b) $Cov[\mathrm{BLUE}(\mathbf{X}\beta)] \prec \mathbf{A} \Leftrightarrow i_-(\mathbf{M}) = r(\mathbf{X}) + n.$

(c) $Cov[\mathrm{BLUE}(\mathbf{X}\beta)] \succcurlyeq \mathbf{A} \Leftrightarrow i_-(\mathbf{M}) = r(\mathbf{X}).$

(d) $Cov[\mathrm{BLUE}(\mathbf{X}\beta)] \preccurlyeq \mathbf{A} \Leftrightarrow i_+(\mathbf{M}) = r[\,\mathbf{X},\,\boldsymbol{\Sigma}\,].$

Corollary

Assume that τ^2 is positive scalar. Then

$$i_+\{\, \tau^2\mathbf{I}_k - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\,\} = i_-\begin{bmatrix} \mathbf{\Sigma} & \tau\mathbf{X} \\ \tau\mathbf{X}' & \mathbf{K}'\mathbf{K} \end{bmatrix} - r(\mathbf{X}) + k,$$

$$i_-\{\, \tau^2\mathbf{I}_k - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\,\} = i_+\begin{bmatrix} \mathbf{\Sigma} & \tau\mathbf{X} \\ \tau\mathbf{X}' & \mathbf{K}'\mathbf{K} \end{bmatrix} - r[\,\mathbf{X},\, \mathbf{\Sigma}\,].$$

If β is estimable, then

$$i_+\{\, \mathbf{I}_p - Cov[\mathrm{BLUE}(\beta)]\,\} = i_-(\, \mathbf{\Sigma} - \mathbf{X}\mathbf{X}'\,),$$
$$i_-\{\, \mathbf{I}_p - Cov[\mathrm{BLUE}(\beta)]\,\} = p + i_+(\, \mathbf{\Sigma} - \mathbf{X}\mathbf{X}'\,) - r[\,\mathbf{X},\, \mathbf{\Sigma}\,].$$

Corollary

Let

$$\mathbf{M} = \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & -\mathbf{A\Sigma A}' & \mathbf{K} \\ \mathbf{X}' & \mathbf{K}' & \mathbf{0} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & -\mathbf{A\Sigma A}' & \mathbf{X} \\ \mathbf{X}' & \mathbf{X}' & \mathbf{0} \end{bmatrix}.$$

Then,

$$i_+\{\, Cov(\mathbf{Ay}) - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\,\} = i_-(\mathbf{M}) - r(\mathbf{X}),$$
$$i_-\{\, Cov(\mathbf{Ay}) - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\,\} = i_+(\mathbf{M}) - r[\,\mathbf{X}, \mathbf{\Sigma}\,].$$

Corollary

$$i_+\{Cov(\mathbf{y}) - Cov[\mathrm{BLUE}(\mathbf{X}\beta)]\}$$
$$= r\{Cov(\mathbf{y}) - Cov[\mathrm{BLUE}(\mathbf{X}\beta)]\} = r[\,\mathbf{X}, \mathbf{\Sigma}\,] - r(\mathbf{X}).$$

If β is estimable, then

$$i_+\{Cov(\mathbf{Ay}) - Cov[\mathrm{BLUE}(\beta)]\} = i_-(\,\mathbf{\Sigma} - \mathbf{XA\Sigma A'X'}\,),$$
$$i_-\{Cov(\mathbf{Ay}) - Cov[\mathrm{BLUE}(\beta)]\} = p + i_+(\,\mathbf{\Sigma} - \mathbf{XA\Sigma A'X'}\,) - r[\,\mathbf{X}, \mathbf{\Sigma}\,].$$

If $E(\mathbf{Ay}) = \mathbf{K}\beta$, then

$$i_+\{Cov(\mathbf{Ay}) - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\}$$
$$= r\{\,Cov(\mathbf{Ay}) - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\} = r[\,\mathbf{\Sigma A'}, \mathbf{X}\,] - r(\mathbf{X}).$$

Inertia formulas for OLSEs' covariance matrices

As is well known, the ordinary least square estimator (OLSE) of an estimable $\mathbf{K}\beta$ ($\mathbf{X}\beta$) under (1) can be written as

$$\mathrm{OLSE}(\mathbf{K}\beta) = \mathbf{K}\mathbf{X}^{+}\mathbf{y}, \quad \mathrm{OLSE}(\mathbf{X}\beta) = \mathbf{X}\mathbf{X}^{+}\mathbf{y} = \mathbf{P}_{\mathbf{X}}\mathbf{y}.$$

In this case,

$$E[\mathrm{OLSE}(\mathbf{K}\beta)] = \mathbf{K}\beta, \qquad Cov[\mathrm{OLSE}(\mathbf{K}\beta)] = \mathbf{K}\mathbf{X}^{+}\boldsymbol{\Sigma}(\mathbf{K}\mathbf{X}^{+})',$$
$$E[\mathrm{OLSE}(\mathbf{X}\beta)] = \mathbf{X}\beta, \qquad Cov[\mathrm{OLSE}(\mathbf{X}\beta)] = \mathbf{P}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{P}_{\mathbf{X}}.$$

It is obvious that if $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}_n$, then $\mathrm{BLUE}(\mathbf{K}\beta) = \mathrm{OLSE}(\mathbf{K}\beta) = \mathbf{K}\mathbf{X}^{+}\mathbf{y}$. Hence, we also have the following.

Theorem

Assume that $\mathbf{\Sigma} = \sigma^2 \mathbf{I}_n$, $\mathbf{K}\beta$ *is estimable. Also let*

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{K} \\ \mathbf{K}' & \mathbf{X}'\mathbf{X} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{A} & \mathbf{X} \\ \mathbf{X}' & \mathbf{X}'\mathbf{X} \end{bmatrix}.$$

Then the following inertia and rank formulas hold

$$i_{+}\{\, \mathbf{A} - Cov[\,\mathrm{OLSE}(\mathbf{K}\beta)]\,\} = i_{+}(\mathbf{M}) - r(\mathbf{X}),$$
$$i_{-}\{\, \mathbf{A} - Cov[\mathrm{OLSE}(\mathbf{K}\beta)]\,\} = i_{-}(\mathbf{M}).$$

If $\mathbf{A} = \mathbf{A}' \in \mathbb{R}^{n\times n}$, *then*

$$i_{+}\{\, \mathbf{A} - Cov[\,\mathrm{OLSE}(\mathbf{X}\beta)]\,\} = i_{+}(\mathbf{N}) - r(\mathbf{X}),$$
$$i_{-}\{\, \mathbf{A} - Cov[\mathrm{OLSE}(\mathbf{X}\beta)]\,\} = i_{-}(\mathbf{N}).$$

The difference of $\mathrm{OLSE}(\mathbf{K}\beta)$ and $\mathrm{BLUE}(\mathbf{K}\beta)$ is

$$\mathrm{OLSE}(\mathbf{K}\beta) - \mathrm{BLUE}(\mathbf{K}\beta) = (\, \mathbf{K}\mathbf{X}^{+} - \mathbf{P}_{\mathbf{K};\mathbf{X};\boldsymbol{\Sigma}})\mathbf{y};$$

The difference of $\mathrm{OLSE}(\mathbf{K}\beta)$ and $\mathrm{BLUE}(\mathbf{K}\beta)$ is

$$\mathrm{OLSE}(\mathbf{K}\beta) - \mathrm{BLUE}(\mathbf{K}\beta) = (\,\mathbf{K}\mathbf{X}^{+} - \mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{\Sigma}})\mathbf{y};$$

the difference of the covariance matrices of $\mathrm{OLSE}(\mathbf{K}\beta)$ and $\mathrm{BLUE}(\mathbf{K}\beta)$ is

$$Cov[\mathrm{OLSE}(\mathbf{K}\beta)] - Cov[\mathrm{BLUE}(\mathbf{K}\beta)] = \mathbf{K}\mathbf{X}^{+}\mathbf{\Sigma}(\mathbf{K}\mathbf{X}^{+})' - \mathbf{P}_{\mathbf{K};\mathbf{X};\mathbf{\Sigma}}\mathbf{\Sigma}\mathbf{P}'_{\mathbf{K};\mathbf{X};\mathbf{\Sigma}};$$

The difference of $\mathrm{OLSE}(\mathbf{K}\beta)$ and $\mathrm{BLUE}(\mathbf{K}\beta)$ is

$$\mathrm{OLSE}(\mathbf{K}\beta) - \mathrm{BLUE}(\mathbf{K}\beta) = (\,\mathbf{K}\mathbf{X}^{+} - \mathbf{P}_{\mathbf{K};\mathbf{X};\boldsymbol{\Sigma}})\mathbf{y};$$

the difference of the covariance matrices of $\mathrm{OLSE}(\mathbf{K}\beta)$ and $\mathrm{BLUE}(\mathbf{K}\beta)$ is

$$Cov[\mathrm{OLSE}(\mathbf{K}\beta)] - Cov[\mathrm{BLUE}(\mathbf{K}\beta)] = \mathbf{K}\mathbf{X}^{+}\boldsymbol{\Sigma}(\mathbf{K}\mathbf{X}^{+})' - \mathbf{P}_{\mathbf{K};\mathbf{X};\boldsymbol{\Sigma}}\boldsymbol{\Sigma}\mathbf{P}'_{\mathbf{K};\mathbf{X};\boldsymbol{\Sigma}};$$

and the covariance matrix of the difference of $\mathrm{OLSE}(\mathbf{K}\beta)$ and $\mathrm{BLUE}(\mathbf{K}\beta)$ is

$$Cov[\,\mathrm{OLSE}(\mathbf{K}\beta) - \mathrm{BLUE}(\mathbf{K}\beta)\,] = (\,\mathbf{K}\mathbf{X}^{+} - \mathbf{P}_{\mathbf{K};\mathbf{X};\boldsymbol{\Sigma}})\boldsymbol{\Sigma}(\,\mathbf{K}\mathbf{X}^{+} - \mathbf{P}_{\mathbf{K};\mathbf{X};\boldsymbol{\Sigma}})'.$$

Theorem

$$
\begin{aligned}
& i_{+}\{\, Cov[\mathrm{OLSE}(\mathbf{K}\beta)] - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\,\} \\
&= r\{\, Cov[\mathrm{OLSE}(\mathbf{K}\beta)] - Cov[\mathrm{BLUE}(\mathbf{K}\beta)]\,\} \\
&= r\begin{bmatrix} \mathbf{\Sigma X} & \mathbf{X} & \mathbf{0} \\ \mathbf{X}'\mathbf{X} & \mathbf{0} & \mathbf{K}' \end{bmatrix} - 2r(\mathbf{X}), \\
& i_{+}\{\, Cov[\mathrm{OLSE}(\mathbf{X}\beta)] - Cov[\mathrm{BLUE}(\mathbf{X}\beta)]\,\} \\
&= r\{\, Cov[\,\mathrm{OLSE}(\mathbf{X}\beta)\,] - Cov[\mathrm{BLUE}(\mathbf{X}\beta)]\,\} \\
&= r[\,\mathbf{\Sigma X}, \,\mathbf{X}\,] - r(\mathbf{X}) < n.
\end{aligned}
$$

If β is estimable, then

$$
\begin{aligned}
& i_{+}\{\, Cov[\mathrm{OLSE}(\beta)] - Cov[\mathrm{BLUE}(\beta)]\,\} \\
&= r\{\, Cov[\mathrm{OLSE}(\beta)] - Cov[\mathrm{BLUE}(\beta)]\,\} \\
&= r[\,\mathbf{\Sigma X}, \,\mathbf{X}\,] - p < n.
\end{aligned}
$$

Formulas for the covariance matrices of residual vectors

Note that two residual vectors with respect to $\mathrm{OLSE}(\mathbf{X}\beta)$ and $\mathrm{BLUE}(\mathbf{X}\beta)$ are given by

$$\hat{\mathbf{e}} = \mathbf{y} - \mathrm{OLSE}(\mathbf{X}\beta) = (\, \mathbf{I}_n - \mathbf{P}_{\mathbf{X}} \,)\mathbf{y} = \mathbf{E}_{\mathbf{X}}\mathbf{y},$$
$$\tilde{\mathbf{e}} = \mathbf{y} - \mathrm{BLUE}(\mathbf{X}\beta) = (\, \mathbf{I}_n - \mathbf{P}_{\mathbf{X};\boldsymbol{\Sigma}} \,)\mathbf{y}.$$

Hence, the covariance matrices of $\hat{\mathbf{e}}$ and $\tilde{\mathbf{e}}$ are given by

$$Cov(\hat{\mathbf{e}}) = E(\hat{\mathbf{e}}\hat{\mathbf{e}}') = \mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}},$$
$$Cov(\tilde{\mathbf{e}}) = E(\tilde{\mathbf{e}}\tilde{\mathbf{e}}') = \boldsymbol{\Sigma} - Cov[\mathrm{BLUE}(\mathbf{X}\beta)].$$

Theorem

Let

$$\mathbf{M} = \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & \mathbf{A} - \mathbf{\Sigma} & \mathbf{X} \\ \mathbf{X}' & \mathbf{X}' & \mathbf{0} \end{bmatrix}$$

Then

$$i_{+}[\,\mathbf{A} - Cov(\tilde{\mathbf{e}})\,] = i_{+}(\mathbf{M}) - r[\,\mathbf{X},\, \mathbf{\Sigma}\,],$$
$$i_{-}[\,\mathbf{A} - Cov(\tilde{\mathbf{e}})\,] = i_{-}(\mathbf{M}) - r(\mathbf{X}).$$

Theorem

In particular,

$$i_{+}[\,\mathbf{I}_n - Cov(\tilde{\mathbf{e}})\,] = i_{+}\begin{bmatrix} \mathbf{\Sigma} - \mathbf{\Sigma}^2 & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{bmatrix} - r[\,\mathbf{X},\, \mathbf{\Sigma}\,] + n,$$

$$i_{-}[\,\mathbf{I}_n - Cov(\tilde{\mathbf{e}})\,] = i_{-}\begin{bmatrix} \mathbf{\Sigma} - \mathbf{\Sigma}^2 & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{bmatrix} - r(\mathbf{X}),$$

$$i_{+}[\,\mathbf{\Sigma} - Cov(\tilde{\mathbf{e}})\,] = r[\,\mathbf{\Sigma} - Cov(\tilde{\mathbf{e}})\,] = r(\mathbf{\Sigma}) + r(\mathbf{X}) - r[\,\mathbf{X},\, \mathbf{\Sigma}\,],$$

$$i_{+}[Cov(\tilde{\mathbf{e}})] = r[Cov(\tilde{\mathbf{e}})] = r[\,\mathbf{X},\, \mathbf{\Sigma}\,] - r(\mathbf{X}).$$

Corollary

$$i_+[\,Cov(\hat{\mathbf{e}}) - Cov(\tilde{\mathbf{e}})\,] = i_-[\,Cov(\hat{\mathbf{e}}) - Cov(\tilde{\mathbf{e}})\,] = r[\,\mathbf{X},\, \mathbf{\Sigma X}\,] - r(\mathbf{X}).$$

Hence, the following statements are equivalent:

(a) $Cov(\hat{\mathbf{e}}) \succcurlyeq Cov(\tilde{\mathbf{e}})$.

(b) $Cov(\hat{\mathbf{e}}) \preccurlyeq Cov(\tilde{\mathbf{e}})$.

(c) $Cov(\hat{\mathbf{e}}) = Cov(\tilde{\mathbf{e}})$.

(d) $r[\,\mathbf{X},\, \mathbf{\Sigma X}\,] = r(\mathbf{X})$.

(e) $\mathscr{R}(\mathbf{\Sigma X}) \subseteq \mathscr{R}(\mathbf{X})$.

(f) $\mathbf{P_X \Sigma} = \mathbf{\Sigma P_X}$.

Comparison of covariance matrices of estimations under two models

Assume that two transformed models of the original model are given as follows

$$\mathscr{M}_1 = \{\, \mathbf{Ay},\ \mathbf{AX}\beta,\ \mathbf{A\Sigma A}' \,\},$$
$$\mathscr{M}_2 = \{\, \mathbf{By},\ \mathbf{BX}\beta,\ \mathbf{B\Sigma B}' \,\},$$

where $\mathbf{A} \in \mathbb{R}^{m_1 \times n}$ and $\mathbf{B} \in \mathbb{R}^{m_2 \times n}$ are two given matrices of arbitrary rank.

Theorem

Let

$$\mathbf{M} = \begin{bmatrix} \mathbf{A\Sigma A'} & 0 & \mathbf{AX} & 0 & 0 \\ 0 & -\mathbf{B\Sigma B'} & 0 & \mathbf{BX} & 0 \\ \mathbf{X'A'} & 0 & 0 & 0 & \mathbf{K'} \\ 0 & \mathbf{X'B'} & 0 & 0 & \mathbf{K'} \\ 0 & 0 & \mathbf{K} & \mathbf{K} & 0 \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{A\Sigma A'} & 0 & \mathbf{AX} \\ 0 & -\mathbf{B\Sigma B'} & \mathbf{BX} \\ \mathbf{X'A'} & \mathbf{X'B'} & 0 \end{bmatrix}.$$

Then

$$i_{+}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\mathbf{K}\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\mathbf{K}\beta)]\,\}$$
$$= i_{+}(\mathbf{M}) - r[\,\mathbf{AX},\ \mathbf{A\Sigma}\,] - r(\mathbf{X}),$$
$$i_{-}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\mathbf{K}\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\mathbf{K}\beta)]\,\}$$
$$= i_{-}(\mathbf{M}) - r[\,\mathbf{BX},\ \mathbf{B\Sigma}\,] - r(\mathbf{X}).$$

In particular, if $\mathbf{X}\beta$ is estimable, then

$$i_{+}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\mathbf{X}\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\mathbf{X}\beta)]\,\} = i_{+}(\mathbf{N}) - r[\,\mathbf{AX},\ \mathbf{A\Sigma}\,],$$
$$i_{-}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\mathbf{X}\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\mathbf{X}\beta)]\,\} = i_{-}(\mathbf{N}) - r[\,\mathbf{BX},\ \mathbf{B\Sigma}\,].$$

If β is estimable, then

$$i_{+}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\beta)]\,\} = i_{+}(\mathbf{N}) - r[\,\mathbf{AX},\ \mathbf{A\Sigma}\,],$$
$$i_{-}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\beta)]\,\} = i_{-}(\mathbf{N}) - r[\,\mathbf{BX},\ \mathbf{B\Sigma}\,].$$

Partition an original model as

$$\mathscr{M} = \left\{ \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \ \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \beta, \ \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{12}' & \boldsymbol{\Sigma}_{22} \end{bmatrix} \right\},$$

where $\mathbf{y}_1 \in \mathbb{R}^{n_1 \times 1}$, $\mathbf{y}_2 \in \mathbb{R}^{n_2 \times 1}$, $\mathbf{X}_1 \in \mathbb{R}^{n_1 \times p}$, $\mathbf{X}_2 \in \mathbb{R}^{n_2 \times p}$, $\boldsymbol{\Sigma}_{11} \in \mathbb{R}^{n_1 \times n_1}$, $\boldsymbol{\Sigma}_{12} \in \mathbb{R}^{n_1 \times n_2}$, $\boldsymbol{\Sigma}_{22} \in \mathbb{R}^{n_2 \times n_2}$. Then two sum-sample models are

$$\mathscr{M}_1 = \{ \mathbf{y}_1, \mathbf{X}_1\beta, \boldsymbol{\Sigma}_{11} \} = \{ \mathbf{A}\mathbf{y}, \mathbf{A}\mathbf{X}\beta, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' \}, \quad \mathbf{A} = [\, \mathbf{I}_{n_1}, \mathbf{0} \,],$$
$$\mathscr{M}_2 = \{ \mathbf{y}_2, \mathbf{X}_2\beta, \boldsymbol{\Sigma}_{22} \} = \{ \mathbf{B}\mathbf{y}, \mathbf{B}\mathbf{X}\beta, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' \}, \quad \mathbf{B} = [\, \mathbf{0}, \mathbf{I}_{n_2} \,].$$

Theorem

Let

$$\mathbf{M} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{0} & \mathbf{X}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{\Sigma}_{22} & \mathbf{0} & \mathbf{X}_2 & \mathbf{0} \\ \mathbf{X}_1' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ \mathbf{0} & \mathbf{X}_2' & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ \mathbf{0} & \mathbf{0} & \mathbf{K} & \mathbf{K} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{0} & \mathbf{X}_1 \\ \mathbf{0} & -\mathbf{\Sigma}_{22} & \mathbf{X}_2 \\ \mathbf{X}_1' & \mathbf{X}_2' & \mathbf{0} \end{bmatrix}.$$

Then

$$\begin{aligned} & i_{+}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\mathbf{K}\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\mathbf{K}\beta)]\,\} \\ & = i_{+}(\mathbf{M}) - r[\,\mathbf{X}_1, \mathbf{\Sigma}_{11}\,] - r(\mathbf{X}_2), \\ & i_{-}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\mathbf{K}\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\mathbf{K}\beta)]\,\} \\ & = i_{-}(\mathbf{M}) - r[\,\mathbf{X}_2, \mathbf{\Sigma}_{22}\,] - r(\mathbf{X}_1). \end{aligned}$$

In particular, if $\mathbf{X}\beta$ is estimable, then

$$i_{+}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\mathbf{X}\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\mathbf{X}\beta)]\,\} = i_{+}(\mathbf{N}) - r[\,\mathbf{X}_1,\, \mathbf{\Sigma}_{11}\,],$$
$$i_{-}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\mathbf{X}\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\mathbf{X}\beta)]\,\} = i_{-}(\mathbf{N}) - r[\,\mathbf{X}_2.\mathbf{\Sigma}_{22}\,],$$

If β is estimable, then

$$i_{+}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\beta)]\,\} = i_{+}(\mathbf{N}) - r[\,\mathbf{X}_1,\, \mathbf{\Sigma}_{11}\,],$$
$$i_{-}\{\, Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\beta)] - Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\beta)]\,\} = i_{-}(\mathbf{N}) - r[\,\mathbf{X}_2,\, \mathbf{\Sigma}_{22}\,].$$

Further work

Possible equalities/inequalities for estimations can be formulated in many different forms. For instance, for the covariance matrices of BLUEs under original and its two sub-sample models, reasonable equalities/inequalities

$$\begin{aligned} & Cov[\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\beta)] \\ & = (\succ, \succcurlyeq, \prec, \preccurlyeq) \frac{1}{2} Cov[\mathrm{BLUE}_{\mathscr{M}_1}(\mathbf{K}\beta)] + \frac{1}{2} Cov[\mathrm{BLUE}_{\mathscr{M}_2}(\mathbf{K}\beta)] \end{aligned}$$

can be proposed, while necessary and sufficient conditions for them to hold can be established by a similarly approach demonstrated previously.

All these results can be used to reveal many profound properties hidden behind estimations.

A general conclusion

For any two symmetric matrix expressions of the same size

$$f(A_1, A_2, \ldots,) \ \text{and} \ g(B_1, B_2, \ldots,),$$

it is always possible to establish certain expansion formulas for calculating the rank/inertia of

$$f(A_1, A_2, \ldots,) - g(B_1, B_2, \ldots,),$$

and to give identifying conditions for

$$f(A_1, A_2, \ldots,) = (\succ, \succcurlyeq, \prec, \preccurlyeq) \ g(B_1, B_2, \ldots,)$$

to hold.

References

- Drygas, H., (1970). The Coordinate-free Approach to Gauss–Markov Estimation. Springer, Heidelberg.
- Isotalo, J., Puntanen, S., Styan, G.P.H., (2008). The BLUE's covariance matrix revisited: a review. J. Statist. Plann. Inference 138, 2722–2737.
- Liski, E.P., Puntanen, S., (1992). Bounds for the trace of the difference of the covariance matrices of the OLSE and BLUE. Linear Algebra Appl. 176, 121–130.
- Puntanen, S., Styan, G.P.H., (1989). The equality of the ordinary least squares estimator and the best linear unbiased estimator, with comments by O. Kempthorne, S.R. Searle, and a reply by the authors. Amer. Statist. 43, 153–164.
- Puntanen, S., Styan, G.P.H., Tian, Y., (2005). Three rank formulas associated with the covariance matrices of the BLUE and the OLSE in the general linear model. Econometric Theory 21, 659–664.

- Puntanen, S., Styan, G.P.H., Werner, H.J., (2000). Two matrix-based proofs that the linear estimator Gy is the best linear unbiased estimator. J. Statist. Plann. Inference 88, 173–179.
- Marsaglia, G., Styan, G.P.H., (1974). Equalities and inequalities for ranks of matrices. Linear Multilinear Algebra 2, 269–292.
- Rao, C.R., (1971). Unified theory of linear estimation. Sankhyā, Ser A 33, 371–394.
- Rao, C.R., (1973). Representations of best linear unbiased estimators in the Gauss–Markoff model with a singular dispersion matrix. J. Multivariate Anal. 3, 276–292.
- Rao, C.R., (1974). Projectors, generalized inverses and the BLUE's. J. Royal Statist. Soc. Ser. B 36, 442–448.
- Rao, C.R., (1976). Estimations of paramters in a linear model. Ann. Statist. 4, 1023–1037.
- Tian, Y., (2010b). Equalities and inequalities for inertias of Hermitian matrices with applications. Linear Algebra Appl. 433, 263–296.
- Tian, Y., (2012a). Solving optimization problems on ranks and

Thank you for your attention!