A Necessary and Sufficient Condition for the Continuity of Local Minima of Parabolic Variational Integrals with Linear Growth

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1. The Equation
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The Parabolic 1-Laplacian

Let E be an open set in $\mathbb{R}^N$, and for $T \in \mathbb{R}^+$ set $E_T = E \times (0, T]$. Consider the quasi-linear, parabolic differential equation

$$u_t - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 \qquad \text{locally in } E_T, \tag{1}$$

and let $(x_o, t_o) \in E_T$.

Problem: What does it take for u to be continuous at (x_o, t_o)?

Motivations - I

Formal limit of the parabolic p-laplacian with $p > 1$:

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0 \qquad \text{locally in } E_T.$$

When $1 < p < 2$, locally bounded solutions are locally Hölder continuous.

- Y. Z. Chen and E. DiBenedetto, Arch. Rational Mech. Anal. (1988);
- Y. Z. Chen and E. DiBenedetto, Arch. Rational Mech. Anal. (1992).

Motivations - II

Geometric meaning:

Consider $v : \Omega \to \mathbb{R}$ and let

$$\Gamma_{v,c} := \{x \in \Omega : \; v(x) = c\}$$

denote a level set of v, which we suppose is a smooth hypersurface, with unit normal $\nu = \frac{Dv}{|Dv|}$. The *mean curvature* of Γ_c is

$$H(\Gamma_{v,c}) = \mathrm{div}(\nu) = \mathrm{div}\left(\frac{Dv}{|Dv|}\right).$$

Therefore, our equation can be rewritten as

$$u_t = H(\Gamma_u).$$

Motivations - III

In order to perform an efficient image reconstruction, Perona and Malik suggest taking the noisy image u_o as initial datum for a diffusion equation such as

$$u_t - \mathrm{div}(a(|Du|^2)Du) = 0,$$

with $a(s)$ positive and decreasing to zero as $s \to \infty$, and under no-flux boundary conditions.
Small diffusion near discontinuities in u_o are supposed to lead to edge preservation, while large diffusion elsewhere would somehow mollify the brightness function and take out noise.

- P. Perona and J. Malik, IEEE Transactions on Pattern Analysis and Machine Intelligence, (1990).

If $a(s) = s^{-1/2}$, then formally the previous equation becomes

$$u_t - \mathrm{div}\left(\frac{Du}{|Du|}\right) = 0,$$

which is commonly called *total variation flow*, TV-flow for short.

- F. Andreu, C. Ballester, V. Caselles, J.M. Mazón and J.S. Moll;
- G. Bellettini and M. Novaga;
- B. Kawohl;
- A. Chambolle and P.L. Lions;
- ...

Motivations - IV

Connection with the Logarithmic Diffusion Equation?

$$u_t - \Delta \ln u = 0$$

This is the formal limit as $m \to 0$ of

$$u_t - \Delta\left(\frac{u^m - 1}{m}\right) = 0.$$

There is a well-known connection between the porous medium and the parabolic p-Laplacian both for $m > 0$, $p > 1$ and $-1 < m < 0$, $p < 1$.

- R.G. Iagar and A. Sánchez, Journal of Mathematical Analysis and Applications, (2008);
- R.G. Iagar, A, Sánchez and J.L. Vázquez, Journal De Mathématiques Pures Et Appliquées, (2007).

The *natural* setting of (1) is the BV space with a proper interpretation of Du as a measure and $\|Du\|$ as its total variation. See

- F. Andreu-Vaillo, V. Caselles and J.-M. Mazón, *Parabolic quasilinear equations minimizing linear growth functionals*, 2004.

It is apparent that in general, no continuity statement can be made for functions in BV. Therefore, we work with a smaller class of solutions.

Notion of Solution

$$u \in C_{\rm loc}\big(0,T;L^2_{\rm loc}(E)\big) \cap L^1_{\rm loc}\big(0,T;W^{1,1}_{\rm loc}(E)\big) \tag{2}$$

Moreover, for every compact set $K \subset E$ and every sub–interval $[t_1, t_2] \subset (0, T]$, u satisfies

$$\int_K u\varphi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \Big[-u\varphi_t + \frac{Du}{|Du|}\cdot D\varphi\Big]\,dxdt = 0, \tag{3}$$

for all bounded testing functions

$$\varphi \in W^{1,2}_{\rm loc}\big(0,T;L^2(K)\big) \cap L^1_{\rm loc}\big(0,T;W^{1,1}_o(K)\big). \tag{4}$$

Here Du is the gradient in the space variable only and, for u in the class (2),

$$\frac{Du}{|Du|} = \begin{cases} \frac{Du}{|Du|} & \text{if } |Du| > 0; \\ 0 & \text{if } |Du| = 0. \end{cases}$$

There Exist Unbounded Solutions

One verifies that any function of the tipe

$$F(|x|,t) = \frac{N-1}{|x|}(at+b) + f(|x|),$$

for $a, b \in \mathbb{R}$ and f an a.e., differentiable function of its argument, is a weak solution of (1)–(4), for $N \geq 3$, provided F_t and $F_{|x|}$ have the same sign. Explicit examples include

$$F(|x|,t) = \frac{N-1}{|x|}(t-1) \qquad \text{for} \quad t \in (0,1),$$

$$F(|x|,t) = \frac{N-1}{|x|}(1-t) + \sin\frac{1}{|x|} \qquad \text{for} \quad t \in (0, \frac{1}{N-1}).$$

One verifies that they satisfy (3), for $N \geq 3$, and yet they are discontinuous at the origin.

Boundedness vs Unboundedness

The existence of unbounded solutions is not a surprise: when $1 < p < \frac{2N}{N+2}$, there exist unbounded solutions to the parabolic p-Laplacian.

On the other hand, one verifies that the function

$$u(x_1, x_2) = \arctan \frac{x_2}{x_1}, \qquad x_1 > 0,$$

is a stationary solution to (1) which is bounded, but discontinuous at the origin. Hence, boundedness is not enough to ensure continuity. This is not a great surprise: from the logarithmic diffusion equation, we already know that extra conditions are required.

Variational Integrals with Linear Growth - I

A hint comes from the *elliptic* setting. Consider the problem

$$\inf_{v\in\mathcal{A}} \int_\Omega \phi(Dv)dx - \int_\Omega fvdx,$$

where $\phi : \mathbb{R}^N \to \mathbb{R}$ is a non-negative convex, sufficiently smooth function satisfying

- $\phi(0) = 0$, $\phi_p(0) = 0$,
- $|p| - \lambda \leq \phi \leq |p|$ for some $\lambda > 0$,
- $\lim_{t\to\infty} \frac{\phi(tp)}{t|p|} = 1$.

$\mathcal{A}$ is a suitable class, Ω is a bounded domain with sufficiently smooth boundary, and f is a given function (for example in L^∞).

A Result by Hardt and Kinderlehrer

An example which is covered by the previous assumptions is

$$\phi(p) = \begin{cases} \frac{1}{2}|p|^2 & \text{for } |p| \le 1 \\ |p| - \frac{1}{2} & \text{for } |p| \ge 1. \end{cases}$$

- R. Hardt and D. Kinderlehrer, Birkhäuser, 1989

prove the following result

Theorem

A local solution u is continuous at a point $x_o \in \Omega$ if and only if

$$\lim_{r\to 0} r \fint_{B_r(x_o)} |Du|dx = 0,$$

where $B_r(y)$ denotes the ball of radius r centered at $y \in E$.

DeGiorgi Classes

A crucial point in Hardt and Kinderlehrer's argument is that local solutions belong to a proper DeGiorgi class, namely they satisfy an inequality of the type

$$\forall y \in \Omega, \qquad \forall B_R(y) \subset \Omega, \qquad \forall 0 < \rho < R, \qquad \forall k \in \mathbb{R},$$

$$\int_{B_\rho(y)} |D(u-k)_\pm| \, dx \le \frac{\gamma}{(R-\rho)} \int_{B_R(y)} |(u-k)_\pm| \, dx + \chi(1 + R^{-1}|k|)|A^\pm_{k,R}|.$$

Here γ, χ are positive constants, $|\Sigma|$ denotes the Lebesgue measure of Σ, and

$$A^\pm_{k,R} \equiv \{x \in B_R(y) : \ (u-k)_\pm > 0\},$$

$$(u-k)_\pm = \{\pm(u-k) \wedge 0\}.$$

Technical Tools

- Just by belonging to DeGiorgi classes, local solutions are locally bounded;
- The condition about the average of the gradient yields continuity, because it implies that the averages $\{\bar{u}_\rho\}$ of the function u in balls of radius ρ form a Cauchy sequence as $\rho \to 0$.

Moreover, the approximate continuity of u plays a fundamental role.

A Suggestion from Hardt and Kinderlehrer

It is not a matter of a PDE, and not even of a variational integral: the point is that u belongs to a proper functional class, namely the DeGiorgi class.

Could it be the case also in the parabolic setting?

Variational Integrals with Linear Growth - II

Let $u \in L^1_{\rm loc}(E_T)$ be such that $u(\cdot, t)$ is of bounded variation in E for $t \in (0, T)$. Assume that the total variation $\|Du(\cdot, t)\|_E$ of the measure $Du(\cdot, t)$, is in $L^1_{\rm loc}(0, T)$ and consider the variational integral

$$\int_0^T \left[\int_E u\varphi_\tau \, dx + \|Du\|_E(\tau)\right] d\tau \le \int_0^T \|D(u+\varphi)\|_E(\tau) \, d\tau \quad (5)$$

for all $\varphi \in C_o^\infty(E_T)$.

Variational Integrals with Linear Growth - II

Local minima of such a functional need not be in the functional class (2) and $Du(\cdot, t)$ is, in general, meant in the sense of measures for a.e. $t \in (0, T)$.
However, if such a minimum is in the class (2), then it is a local, weak solution of to the 1-Laplacian equation (1) by

- V. Bögelein, F. Duzaar and P. Marcellini, *SIAM J. Imaging Sci., to appear;*
- V. Bögelein, F. Duzaar, J. Kinnunen and P. Marcellini, *personal communication*.

Hence, a necessary and sufficient condition for a locally bounded u to be continuous at some $(x_o, t_o) \in E_T$ continues to hold for minima of parabolic variational integrals of the form (5).

Singular Parabolic DeGiorgi Classes - I

Introduce the cylinders

$$Q_\rho(\theta) = B_\rho \times (-\theta\rho, 0],$$

where θ is a positive parameter to be chosen as needed. If $\theta = 1$ we write Q_ρ. For a point $(y, s) \in \mathbb{R}^{N+1}$ we let $[(y, s) + Q_\rho(\theta)]$ be the cylinder of "vertex" at (y, s) and congruent to $Q_\rho(\theta)$, i.e.

$$[(y, s) + Q_\rho(\theta)] = B_\rho(y) \times (s - \theta\rho, s].$$

Let $\mathcal{C}\big(Q_\rho(\theta)\big)$ denote the class of all non-negative, piecewise smooth, cutoff functions ζ defined in $Q_\rho(\theta)$, vanishing outside B_ρ, such that $\zeta_t \geq 0$ and satisfying

$$|D\zeta| + \zeta_t \in L^\infty\big(Q_\rho(\theta)\big).$$

Singular Parabolic DeGiorgi Classes - II

The singular parabolic DeGiorgi class, $[DG](E_T;\gamma)$, is the collection of all u in the functional class (2), satisfying

$$
\begin{aligned}
\sup_{s-\theta\rho\le t\le s}\int_{B_\rho(y)}&(u-k)_\pm^2\zeta(x,t)dx\\
&+\iint_{[(y,s)+Q_\rho(\theta)]}|D(u-k)_\pm|\zeta dxd\tau\\
&\le\gamma\iint_{[(y,s)+Q_\rho(\theta)]}\Big[(u-k)_\pm|D\zeta|+(u-k)_\pm^2|\zeta_\tau|\Big]dxd\tau\\
&\qquad+\int_{B_\rho(y)}(u-k)_\pm^2\zeta(x,s-\theta\rho)dx
\end{aligned}
$$

for a positive constant γ, for all cylinders $[(y,s)+Q_\rho(\theta)]\subset E_T$, all $k\in\mathbb{R}$, and all $\zeta\in\mathcal{C}\big([(y,s)+Q_\rho(\theta)]\big)$.

The General Perspective

Proposition

Let u be a local, weak solution to (1) in the sense of (2)–(4). Then $u \in [DG](E_T;\gamma)$, for $\gamma = 2$.

Proposition

Let u in the class (2) be a minimum of the parabolic variational integral (5). Furthermore, assume that $u_t \in L^2(E_T)$. Then $u \in [DG](E_T;\gamma)$, for some $\gamma>0$.

We have followed a typical path, namely

PDE $\Rightarrow$ Minima of Variational Integral $\Rightarrow$ DeGiorgi Classes

Theorem

Let $u \in [DG](E_T;\gamma)$, for some $\gamma>0$, and assume that it is locally bounded. Then, u is continuous at some $(x_o,t_o) \in E_T$, if and only if

$$\limsup_{\rho\searrow 0} \rho \iint_{[(x_o,t_o)+Q_\rho]} |Du|\,dx d\tau = 0. \tag{6}$$

Remark

- The theorem gives a necessary and sufficient condition for continuity at a single given point, not in a neighborhood.
- Condition (6) cannot be replaced by the weaker condition
  $$\limsup_{\rho\searrow 0} \rho \iint_{[(x_o,t_o)+Q_\rho]} |Du|\,dx d\tau = \alpha > 0.$$
- We rely only on the fact that $u \in [DG](E_T;\gamma)$, for some $\gamma>0$.
- (6) is the natural generalization of Hardt-Kinderlehrer's condition

A Remark about the Modulus of Continuity

Condition (6) provides no information on the modulus of continuity of u at (x_o, t_o).
Consider the two stationary solutions of (1)–(4), in a neighborhood of the origin of $\mathbb{R}^2$,

$$u_1(x_1, x_2) = \begin{cases} \dfrac{1}{\ln x_1} & \text{for } x_1 > 0 \\ 0 & \text{for } x_1 \le 0; \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} \sqrt{x_1} & \text{for } x_1 > 0 \\ 0 & \text{for } x_1 \le 0. \end{cases}$$

They can be regarded as equibounded near the origin. They both satisfy (6), and exhibit quite different moduli of continuity at the origin.

This occurrence is in line with a remark of Evans:

- L.C. Evans, Contemp. Math. (2007).

A sufficiently smooth solution of the elliptic 1-Laplacian equation is a function whose level sets are surfaces of zero mean curvature.
Thus, if u is a solution, so is $\varphi(u)$ for all continuous monotone functions $\varphi(\cdot)$. This implies that a modulus of continuity cannot be identified solely in terms of an upper bound of u.

A Heuristic Justification

Suppose that in

$$\sup_{-\theta\rho\le t\le 0}\int_{B_\rho}(u-k)_\pm^2\zeta(x,t)dx+\iint_{Q_\rho(\theta)}|D(u-k)_\pm|\zeta dxd\tau$$
$$\le\gamma\iint_{Q_\rho(\theta)}\Big[(u-k)_\pm|D\zeta|+(u-k)_\pm^2|\zeta_\tau|\Big]dxd\tau$$
$$+\int_{B_\rho}(u-k)_\pm^2\zeta(x,-\theta\rho)dx$$

we can estimate

$$\iint_{Q_\rho(\theta)}|D(u-k)_\pm|\zeta dxd\tau\ge\gamma\frac{\rho^{p-1}}{\omega^{p-1}}\iint_{Q_\rho(\theta)}|D(u-k)_\pm|^p\zeta^p dxd\tau$$

where ω is the oscillation of u in $Q_\rho(\theta)$.

Then, we would obtain

$$
\begin{aligned}
&\sup_{-\theta\rho\le t\le 0}\int_{B_\rho}(u-k)_\pm^2\zeta(x,t)dx \\
&\qquad + \gamma\frac{\rho^{p-1}}{\omega^{p-1}}\iint_{Q_\rho(\theta)}|D(u-k)_\pm|^p\zeta^p dxd\tau \\
&\qquad \le \gamma\iint_{Q_\rho(\theta)}\left[(u-k)_\pm|D\zeta|+(u-k)_\pm^2|\zeta_\tau|\right]dxd\tau \\
&\qquad\qquad + \int_{B_\rho}(u-k)_\pm^2\zeta(x,-\theta\rho)dx,
\end{aligned}
$$

and by known techniques, u would be locally Hölder continuous.

On the other hand, by the Hölder inequality,

$$\iint_{Q_\rho(\theta)} |D(u-k)_\pm|^p \zeta^p dx d\tau \geq \left[\iint_{Q_\rho(\theta)} |D(u-k)_\pm| \zeta dx d\tau \right]^p .$$

Combining the previous inequalities, yields

$$\gamma\omega \geq \rho \iint_{Q_\rho(\theta)} |D(u-k)_\pm| \zeta dx d\tau.$$

A Possible Generalization - I

Consider quasi-linear evolutions equations of the type

$$u \in C_{\mathrm{loc}}(0, T; L^2_{\mathrm{loc}}(E)) \cap L^1_{\mathrm{loc}}(0, T; W^{1,1}_{\mathrm{loc}}(E)) \tag{7}$$

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, Du) = B(x, t, u, Du) \quad \text{weakly in } E_T, \tag{8}$$

where the functions $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $B : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ are only assumed to be measurable and subject to the structure conditions

$$\begin{aligned} C_o|Du| - C \leq & \mathbf{A}(x, t, u, Du) \cdot Du \leq C_1|Du| + C \\ & |\mathbf{A}(x, t, u, Du)| \leq C(1 + |u|) \\ & |B(x, t, u, Du)| \leq C(1 + |u|) \end{aligned} \tag{9}$$

for given positive constants $C_o \leq C_1$ and C.

A Possible Generalization - II

The first of (9) does not insure, in general that the equation in (8) is parabolic. For example the vector field

$$\mathbf{A}(x,t,u,Du) = \frac{Du}{|Du|}\Big(1 - \frac{1}{|Du|}\Big)$$

satisfies the first of (9) but its modulus of ellipticity changes sign at $|Du| = 1$.

Parabolicity is insured if one requires that **A** is such that the truncations $\pm(u-k)_\pm$ are sub-solutions of (8). This in turn is insured if, in addition to (9) one requires that

$$\mathbf{A}(x,t,\xi,\eta)\cdot\eta \geq 0$$

What about Boundedness?

Proposition

Let $u \in [DG]^{\pm}(E_T, \gamma)$, let $r > N$ and assume that $u \in L^r_{loc}(E_T)$. Then, there exists a positive constant γ_o depending only upon N, γ, r, such that

$$\sup_{K_\rho(y)\times[s,t]} u_\pm \le \gamma_o \left(\frac{\rho}{t-s}\right)^{\frac{N}{r-N}} \left(\fint_{2s-t}^{t} \fint_{K_{4\rho}(y)} u_\pm^r \, dx d\tau\right)^{\frac{1}{r-N}}$$
$$+ \gamma_o \frac{t-s}{\rho}$$

for all cylinders

$$K_{4\rho}(y) \times [s-(t-s), s+(t-s)] \subset E_T.$$

The constant $\gamma_o(N, \gamma, r) \to \infty$ as either $r \to \max\{1; N\}$, or $r \to \infty$.

Future Work: Open Problems

For $p > 1$ a local upper bound on $|u_p|$ suffices to establish that u_p is locally Hölder continuous in E_T, with Hölder constant and exponent depending on such a local upper bound and p.

However, even if $|u_p|$ is locally bounded in E_T, uniformly in p, the Hölder constants and exponents deteriorate as $p \searrow 1$, in line with the previous remarks.

Because of this structural difference between p-Laplacian and 1-Laplacian equation, a topology by which (1) can be identified as a rigorous limit of the parabolic p-Laplacian with $p > 1$ represents a challenging problem.

Phrased in a different way: if bounded solutions for $p > 1$ are Hölder continuous and for $p = 1$ may have any modulus of continuity, in which sense can we talk about stability?

What is the connection between the condition that ensures continuity for solutions to (1) and the condition that gives the analogous result for solutions to the logarithmic diffusion equation?

For the moment ...

Thank you for your attention!