A semilinear elliptic problem with a singularity at $u = 0$

joint work by

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inspired by papers by

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<u>The problem</u> : Find u

$$\begin{cases} u \geqslant 0 \text{ on } \Omega, \\ -\operatorname{div} A(x)\, Du = F(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ open, bounded

$A(x) \in L^\infty(\Omega)^{N\times N}$, $A(x) \geqslant \alpha I$, $\alpha > 0$,

$0 \leqslant F(x,s) \leqslant h(x)\left(1 + \dfrac{1}{s^\gamma}\right)$ $\forall s > 0$ a.e. $x \in \Omega$,

$F : \Omega \times [0,+\infty[\longrightarrow [0,+\infty]$ Carathéodory,

$h \in L^{(2^*)'}(\Omega)$, $\gamma > 0$.

Existence?
Stability?
Uniqueness?

Definition of the solution?

$\boxed{\text{The case } 0 < \gamma \leq 1}$: Easy

<u>Definition of the solution</u>

u is a solution iff

$$\begin{cases} u \in H_0^1(\Omega), \\ u \geq 0, \end{cases}$$

$$\begin{cases} \forall v \in H_0^1(\Omega),\ v \geq 0, \\ \int_\Omega F(x,u)\, v < +\infty, \\ \int_\Omega A\, Du\, Dv = \int_\Omega F(x,u)\, v. \end{cases}$$

This is a "very natural" definition ...

... and it works!

<u>Proof of existence</u>

Let $F_\varepsilon(x,s) = T_{1/\varepsilon}(F(x,s^+))$

Then $\exists\, u_\varepsilon$

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega), \\ -\operatorname{div} A(x) D u_\varepsilon = F_\varepsilon(x,u_\varepsilon) \text{ in } \mathcal{D}'(\Omega), \end{cases}$$

$[\Rightarrow u^\varepsilon \geqslant 0]$

<u>Test</u> u_ε

$$\int_\Omega A D u_\varepsilon D u_\varepsilon = \int_\Omega F_\varepsilon(x,u_\varepsilon)\, u_\varepsilon$$

$$\leqslant \int_\Omega h(x)\left(1 + \frac{1}{u_\varepsilon^\gamma}\right) u_\varepsilon$$

$$\leqslant \int_\Omega h(x)\, C_\gamma \left(1 + \frac{1}{u_\varepsilon}\right) u_\varepsilon$$

$$= C'_\gamma \int_\Omega h(u_\varepsilon + 1)$$

Therefore $\|u_\varepsilon\|_{H_0^1(\Omega)} \leq C$

$u_\varepsilon \longrightarrow u \ H_0^1(\Omega)$ weak and a.e. $x \in \Omega$

Test $v \in H_0^1(\Omega)$, $v \geq 0$

$$\int_\Omega A\, Du_\varepsilon\, Dv = \int_\Omega F_\varepsilon(x, u_\varepsilon)\, v$$

$$\downarrow \qquad \text{Fatou} \qquad F_\varepsilon(x,u_\varepsilon) \to F(x,u)$$

$$\int_\Omega A\, Du\, Dv \geq \int_\Omega F(x,u)\, v$$

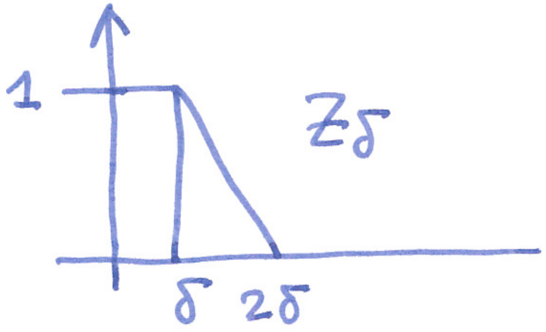
Therefore $\forall v \in H_0^1(\Omega)$, $v \geq 0$

$$\int_\Omega F(x,u)\, v < +\infty$$

$\geq$? or $=$?

<u>New test function</u> : $Z_\delta(u_\varepsilon)\,\varphi$ $\quad \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$

$\varphi \geq 0$



$$\int_\Omega A\,Du_\varepsilon\,D\varphi\,Z_\delta(u_\varepsilon) + \overbrace{\int_\Omega A\,Du_\varepsilon\,Du_\varepsilon\,Z'_\delta(u_\varepsilon)\,\varphi}^{\leq 0} =$$

$$= \int_\Omega F_\varepsilon(x,u_\varepsilon)\,Z_\delta(u_\varepsilon)\,\varphi \geq \int_{\{u_\varepsilon \leq \delta\}} F_\varepsilon(x,u_\varepsilon)\,\varphi$$

δ fixed , $\varepsilon \to 0$

$$\int_\Omega A\,Du\,D\varphi\,Z_\delta(u) \geq \limsup_\varepsilon \int_{\{u_\varepsilon \leq \delta\}} F_\varepsilon(x,u_\varepsilon)\,\varphi$$

$\downarrow \delta \to 0$

$$\int A\,Du\,D\varphi\,\chi_{\{u=0\}} = 0$$

Passing to the limit $\quad \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)\,,\ \varphi \geq 0$

$$\int_\Omega A Du_\varepsilon D\varphi = \int_\Omega F_\varepsilon(x,u_\varepsilon)\varphi = \underbrace{\int_{\{u_\varepsilon \leq \delta\}} F_\varepsilon(x,u_\varepsilon)\varphi}_{I_\varepsilon^\delta} + \underbrace{\int_{\{u_\varepsilon > \delta\}} F_\varepsilon(x,u_\varepsilon)\varphi}_{II_\varepsilon^\delta}$$

$$\limsup_\varepsilon I_\varepsilon^\delta \leq \omega(\delta) \to 0 \text{ if } \delta \to 0$$

$$II_\varepsilon^\delta : \quad 0 \leq F_\varepsilon(x,u_\varepsilon)\varphi\,\chi_{\{u_\varepsilon>\delta\}} \leq h(x)\left(1+\frac{1}{u_\varepsilon^\gamma}\right)\chi_{\{u_\varepsilon>\delta\}}\varphi \leq$$

$$\Big\downarrow \varepsilon \text{ a.e.} \qquad \leq h(x)\left(1+\frac{1}{\delta^\gamma}\right)\varphi \in L^1(\Omega)$$

$$F(x,u)\varphi\,\chi_{\{u>\delta\}} \quad \dots \text{ if } \operatorname{meas}\{u=\delta\}=0 \;!!$$

$$\int_\Omega F(x,u)\varphi\,\chi_{\{u>\delta\}} \xrightarrow{\delta} \int_\Omega F(x,u)\varphi\,\chi_{\{u>0\}}$$

$$\int_\Omega A Du D\varphi = \int_{\{u>0\}} F(x,u)\varphi \quad \forall \varphi \in H_0^1(\Omega)\cap L^\infty(\Omega),\ \varphi \geq 0$$

But $\displaystyle\int_{\{u=0\}} F(x,u)\,\varphi = 0$

because

$$\int_\Omega A\,Du\,D\varphi\; Z_\delta(u) \geq \limsup_\varepsilon \int_{\{u_\varepsilon \leq \delta\}} F_\varepsilon(x,u_\varepsilon)\,\varphi \geq$$

$$\geq \liminf_\varepsilon \int_{\{u_\varepsilon \leq \delta\}} F_\varepsilon(x,u_\varepsilon)\,\varphi\,.$$

$$\geq (\text{Fatou}) \int_{\{u \leq \delta\}} F(x,u)\,\varphi \quad \text{if } \operatorname{meas}\{u=\delta\}=0$$

$$\geq \int_{\{u=0\}} F(x,u)\,\varphi$$

$\downarrow$ 0 if $\delta \to 0$

Therefore $\displaystyle\int_\Omega A\,Du\,D\varphi = \int_\Omega F(x,u)\,\varphi \quad \forall \varphi \in H^1_0(\Omega)\cap L^\infty(\Omega)$, $\varphi \geq 0$

$\Rightarrow \forall \varphi \in H^1_0(\Omega)$.

When $0 < \gamma \leq 1$

Existence is proved

Actually stability is proved:

if $0 \leq F_\varepsilon(x,s) \leq h(x)\left(1+\frac{1}{s^\gamma}\right)$

$$F_\varepsilon(x,s_\varepsilon) \longrightarrow F(x,s) \quad \text{a.e. } x \in \Omega$$

if $s_\varepsilon \to s$ $\quad s_\varepsilon \geq 0,\ s \geq 0$

Uniqueness ?

OK if $F(x,s)$ non increasing on s.

$$(F(x,s_1) - F(x,s_2))(s_1 - s_2) \leq 0.$$

$\boxed{\text{The case } \gamma > 1}$ is more complicated

The <u>method</u> is the same

but the <u>definition</u> of the solution

is much more complicated

The <u>result</u> is the same ....

... with another definition !

Definition for $\gamma > 1$:

$$
\begin{cases}
u \in H^1_{loc}(\Omega), \\
u \geq 0, \\
G_k(u) \in H^1_0(\Omega) \quad \forall k > 0, \\
T_k(u)^{\frac{\gamma+1}{2}} \in H^1_0(\Omega) \quad \forall k > 0, \\
\varphi Du \in L^2(\Omega)^N \quad \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega).
\end{cases}
$$

$\forall v \in \mathcal{V}(\Omega)$, $v \geq 0$,

ie $\forall v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $v \geq 0$,

s.t. $\exists\, \hat{f} \in L^1(\Omega)$, $\exists\, \hat{\varphi}_i \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\exists\, \hat{g}_i \in L^2(\Omega)^N$,

with $-\operatorname{div} {}^tA(x) Dv = \hat{f} + \sum_i \hat{\varphi}_i (-\operatorname{div} \hat{g}_i)$ in $\mathcal{D}'(\Omega)$,

Then

i) $\int_\Omega F(x,u)\, v < +\infty$,

ii) "$\int_\Omega (-\operatorname{div} A(x) Du)\, v$" = "$\int_\Omega (-\operatorname{div} {}^tA(x) Dv)\, u$" =

$$= \langle -\operatorname{div} {}^tA(x) Dv, G_k(u) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} +$$

$$+ \int_\Omega \hat{f}\, T_k(u) + \sum_i \int_\Omega \hat{g}_i\, D(\hat{\varphi}_i T_k(u)) =$$

$$= \int_\Omega F(x,u)\, v.$$