Quasiminimizers – definitions, constructions, and capacity estimates

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1 Introduction

Quasiminimizers have been previously used as tools in studying regularity of minimizers of variational integrals. However, quasiminimizers cover a wide range of applications and their properties are based only on the minimization of the variational integrals instead of the corresponding Euler equation. For example, regularity properties as Hölder continuity and L^p–estimates are consequences of the quasiminimizing property. The theory of quasiminimizers can be easily extended to metric measure spaces since the theory uses the absolute value of the gradient and not more subtle properties of the derivative, see [KM1], [KiM] and [Sh1–2]. From the theory of quasiminimizers one can also learn which properties of p–harmonic functions and other potential functions are stable under perturbations of (small) energy changes.

From the potential theoretic point of view quasiminimizers have several drawbacks:

- no unique solution for the Dirichlet problem
- no comparison principle
- quasiminimizers do not form a sheaf
- no linearity

Instead of using quasiminimizers as tools, the objective of these lectures is to show that quasiminimizers have a fascinating theory themselves. In particular, they form a basis for nonlinear potential theoretic model with interesting features. The following aspects of quasiminimizers are considered:

- Definitions
- Quasiminimizers in $\mathbf{R}$
- Constructions for quasiminimizers
- Capacity estimates
- Open problems

In the preliminary Section 2 some basic concepts of Sobolev spaces and capacity are explained.

2 Preliminaries

We use the standard definition for the first order Sobolev space $W^{1,p}(\Omega)$. We recall some properties of this space. Let Ω be an open set in $\mathbf{R}^n$ and $1 \leq p < \infty$. A (real valued) function v belongs to $W^{1,p}(\Omega)$ if and only if $v \in L^p(\Omega)$ and its distributional partial derivatives $\partial_i v$, $i = 1, ..., n$, belong to $L^p(\Omega)$. We let $\nabla v = (\partial_1 v, ..., \partial_n v)$ denote the distributional gradient of v. Being the distributional gradient of v means that

$$\int_\Omega v \nabla \varphi \, dx = - \int_\Omega \nabla v \varphi \, dx$$

holds for all $\varphi \in C_0^1(\Omega)$. The space $W^{1,p}(\Omega)$ is a Banach space under the norm

$$\| v \|_{W^{1,p}(\Omega)} = \| v \|_{1,p} = \| v \|_p + \| \, |\nabla v| \, \|_p \, .$$

Here $\| v \|_p = \| v \|_{L^p(\Omega)}$ refers to the standard L^p–norm

$$\left(\int_\Omega |v|^p \, dx\right)^{1/p}$$

of a function v in Ω. The space $W^{1,p}_{loc}(\Omega)$ consists of all functions v such that $v|\Omega'$ belongs to $W^{1,p}(\Omega')$ for all open $\Omega' \subset\subset \Omega$, i.e. $\overline{\Omega'}$ is a compact subset of Ω. The space $C^1(\Omega)$ as well the space of locally lipschitz functions are dense in $W^{1,p}_{loc}(\Omega)$.

The first order Sobolev space space $W^{1,p}(\Omega)$ is also a lattice, i.e. $\max(u, v)$ and $\min(u, v)$ belong to $W^{1,p}(\Omega)$ whenever $u, v \in W^{1,p}(\Omega)$.

The concept of capacity plays an important role in the theory of Sobolev spaces and Potential Theory. If C is a compact subset of Ω, then the pair $E = (C, \Omega)$ is called a *condenser*. The *p–capacity* of E is defined as

$$cap_p E = \inf \int_\Omega |\nabla \varphi|^p \, dx \tag{1}$$

where the infimum is taken over all functions $\varphi \in C_o^\infty(\Omega)$ such that $\varphi = 1$ on C. There is a standard way to extend the concept of the p–capacity first to condensers (U, Ω) where U is an open set in Ω and then to condensers (V, Ω) where V is an arbitrary subset of Ω, see [HKM, Chapter 2]. A set V is of p–capacity zero if $cap_p(V, \Omega) = 0$ for all open sets Ω. Sets of p–capacity zero have a similar role in the theory of Sobolev spaces as sets of measure zero in the theory of L^p–spaces. Note that for $1 < p < n$ a set $V \subset \mathbf{R}^n$ of p–capacity zero has Hausdorff dimension at most $n - p$. For $p > n$ such a set is empty and for $p = n$ all Hausdorff $\alpha > 0$ measures of V are zero. The case $n = 1$ is somewhat exceptional: for $p \geq 1$ all sets of p–capacity zero are empty.

There is a closely related capacity $Cap_p(C)$, called the p–Sobolev capacity defined as

$$Cap_p(C) = inf \parallel v \parallel_{1,p,\mathbf{R}^n}$$

where the infimum is taken over all functions $v \in C^1(\mathbf{R}^n)$ such that $v = 1$ on C. It has the advantage that there is no open set (except $\mathbf{R}^n$) which includes C. As above this definition is extended to an arbitrary subset V of $\mathbf{R}^n$. Sets of p–capacity zero are the same under both definitions although the values of capacities $cap_p(V, \Omega)$ and $Cap_p(V)$ are different in most cases.

The concept of the capacity makes it possible to give a more precise interpretation for functions in $W^{1,p}(\Omega)$ than just equivalence classes in $L^p(\Omega)$: a function $v \in W^{1,p}(\Omega)$ has a p–quasicontinuous version v_o. This means that $v = v_o$ a.e. in Ω and for each $\varepsilon > 0$ there is a closed set $C \subset \Omega$ such $v_o|C$ is continuous and $Cap_p(\Omega \setminus C) < \varepsilon$. A continuous function $v \in W^{1,p}(\Omega)$ is clearly quasicontinuous. It can be shown that if u and v are p–quasicontinuous and $u = v$ a.e. in Ω, then $u = v$ except in a set of p–capacity zero. This is abbreviated as $u = v$ p–q.e. Hence functions in $W^{1,p}(\Omega)$ are essentially defined pointwise up to a set of p–capacity zero. See [HKM, Chaper 4]. Also a function $v \in W^{1,p}_{loc}(\Omega)$ has a so called ACL^p–representative (ACL = absolutely continuous on lines).

The space $W_0^{1,p}(\Omega)$ consists of all functions $v \in W^{1,p}(\Omega)$ such that v can be approximated in the norm $\parallel v \parallel_{1,p}$ by functions in the class $C_0^1(\Omega)$. The space $W_0^{1,p}(\Omega)$ is a closed subspace of $W^{1,p}(\Omega)$. Every function $v \in W_0^{1,p}(\Omega)$ can be extended by 0 to $\mathbf{R}^n \setminus \Omega$ and the extended function belongs to $W^{1,p}(\mathbf{R}^n) = W_0^{1,p}(\mathbf{R}^n)$. Conversely, a p–quasicontinuous function $v \in W^{1,p}(\mathbf{R}^n)$ belongs to $W_0^{1,p}(\Omega)$ if (and only if) $v = 0$ p–q.e. in $\partial\Omega$. Note that a continuous function $v \in W_0^{1,p}(\Omega)$ need not have boundary values 0 on $\partial\Omega$. Boundary behavior of v depends on the capacitary properties of $\partial\Omega$.

Capacity can be used to define the space $W_0^{1,p}(E)$ for an arbitrary subset E of $\mathbf{R}^n$, namely $W_0^{1,p}(E)$ consists of all p–quasicontinuous functions in $W^{1,p}(\mathbf{R}^n)$ such that $u = 0$ p–q.e. in $\mathbf{R}^n \setminus E$. This definition is useful only if E is measurable. It can be used, for example, if the set E is a level set of a function in $W^{1,p}(\mathbf{R}^n)$.

Sobolev spaces on the real line are rather simple. If (a,b), $-\infty \le a < b \le \infty$ is an open interval in $\mathbf{R}$, then the space $W^{1,p}((a,b))$ consists of $L^p(a,b)$–functions v which are absolutely continuous in each closed interval $[c,d] \subset (a,b)$ with $v' \in L^p(a,b)$. This requires v to be a quasicontinuous version. If (a,b) is a finite interval, then the function v has a continuous extension to the endpoints and v is absolutely continuous in $[a,b]$. A function $v \in W_0^{1,p}((a,b))$, (a,b) a finite interval, can be characterized by the properties: (i) v is absolutely continuous on $[a,b]$, (ii) $v(a) = 0 = v(b)$ and (iii) $v' \in L^p(a,b)$.

3 Definitions

Let Ω be an open subset of $\mathbf{R}^\mathbf{n}$, $n \ge 1$, $p > 1$ and $K \ge 1$. A function u in the local Sobolev space $W_{loc}^{1,p}(\Omega)$ is called a (p,K)–*quasiminimizer* in Ω if for all open sets $\Omega' \subset\subset \Omega$

$$\int_{\Omega'} |\nabla u|^p \, dx \le K \int_{\Omega'} |\nabla v|^p \, dx \tag{2}$$

for all functions v such that $v - u \in W_0^{1,p}(\Omega')$. In general we keep the number p fixed and use the abbreviation K–quasiminimizer. For $K = 1$ the function u is minimizer and hence a p–harmonic function. This means that a function $u \in W_{loc}^{1,p}(\Omega)$ is a 1–quasiminimizer if and only if u is a weak solution of the Euler equation

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = 0 \tag{3}$$

of the variational integral

$$\int_\Omega |\nabla u|^p \, dx.$$

Here (3) is supposed to hold for all $\varphi \in C_o^1(\Omega)$. If $u \in C^2(\Omega)$, then (3) can be written in the form

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0. \tag{4}$$

However, solutions of (3) are, in general, in $C^{1,\alpha}$ only.

More general sets than open sets can be used in (2). If E is a measurable set with compact closure in Ω, then

$$\int_E |\nabla u|^p\,dx \le K \int_E |\nabla v|^p\,dx \tag{5}$$

for every v with $v-u \in W_0^{1,p}(E)$. See [KM].

Example 3.1 Every (weak) solution u of the equation

$$\nabla \cdot A(x, \nabla u) = 0$$

where the operator A satisfies

$$\alpha|h|^p \le A(x,h)\cdot h \le \beta|h|^p,\ 0<\alpha\le\beta<\infty,$$

is a K–quasiminimizer with $K=(\beta/\alpha)^p$, see [HKM, p. 59].

Next we introduce an alternative definition for a quasiminimizer. In particular this is useful for quasiminimizers on the real line. The definition does not involve the p–Dirichlet integral of the quasiminimizer u itself but only that of minimizers, i.e. solutions of the p–harmonic equation with the same boundary values as u.

Let $u \in W^{1,p}_{loc}(\Omega), p>1$. For each open set $\Omega' \subset\subset \Omega$ we let $u_{\Omega'}$ denote the minimizer of the p–Dirichlet integral in Ω' with boundary values u, i.e. $u_{\Omega'} - u \in W_0^{1,p}(\Omega')$ and $u_{\Omega'}$ is a solution of the p–harmonic equation (4) in Ω'. Direct methods in the calculus of variations are used to show that such a unique minimizer always exists, see [HKM, Chapter 5]. Condition (2) can now be rewritten as

$$\int_{\Omega'} |\nabla u|^p\,dx \le K \int_{\Omega'} |\nabla u_{\Omega'}|^p\,dx. \tag{6}$$

Theorem 3.2 *Suppose that u belongs to $W^{1,p}_{loc}(\Omega)$. Then u is a K–quasiminimizer in Ω if and only if for each open set $\Omega' \subset\subset \Omega$ and all disjoint open sets $\Omega_1, ..., \Omega_k \subset \Omega'$ it holds*

$$\sum_i \int_{\Omega_i} |\nabla u_{\Omega_i}|^p\,dx \le K \int_{\Omega'} |\nabla u_{\Omega'}|^p\,dx. \tag{7}$$

Proof. The necessity of condition (7) is immediate.

For the converse we have to show (6) in every open set $\Omega' \subset\subset \Omega$. Fix an open set Ω' and note that (7) holds for a countable collection of open subsets $\Omega_1, \Omega_2, ...$ of Ω' as well. Form a Whitney decomposition $\{Q_i\}$ of Ω' where the open cubes Q_i are disjoint and $\cup\overline{Q_i} = \Omega'$. For each $j = 1, 2, ...$

subdivide every cube Q_i, if necessary, to a finite number of disjoint cubes to obtain a new sequence $\{Q_i^j\}$ of disjoint cubes so that every cube Q_i^j satisfies $diam(Q_i^j) \leq 1/j$.

Next define for each j the function v_j as

$$\begin{aligned} v_j(x) &= u_{Q_i^j}(x), x \in Q_i^j,\ i = 1, 2, ... \\ &= u(x), x \in \Omega \setminus \cup_i Q_i^j. \end{aligned}$$

Now it easily follows that $v_j - u \in W_0^{1,p}(\Omega')$ and by (7)

$$\int_{\Omega'} |\nabla v_j|^p \, dx \leq K \int_{\Omega'} |\nabla u_{\Omega'}|^p \, dx \tag{8}$$

for each j. Since the sequence ∇v_j is bounded in $L^p(\Omega')$ and $v_j - u \in W_0^{1,p}(Q_i^j)$ for each i and j, the Sobolev inequality yields

$$\int_{Q_i^j} |v_j - u|^p \, dx \leq C diam(Q_i^j)^p \int_{Q_i^j} |\nabla (v_j - u)|^p \, dx$$

$$\leq C j^{-p} \int_{Q_i^j} |\nabla (v_j - u)|^p \, dx$$

where C depends only on p and n. Summing over i we obtain

$$\int_{\Omega'} |v_j - u|^p \, dx \leq \sum_i \int_{Q_i^j} |v_j - u|^p \, dx \leq 2^{p+1} C j^{-p} \int_{\Omega'} |\nabla u|^p \, dx$$

because

$$\int_{\Omega'} |\nabla v_j|^p \, dx \leq \int_{\Omega'} |\nabla u|^p \, dx$$

by he minimizing property of the function v_j in each Q_i^j. Thus $v_j \to u$ in $L^p(\Omega')$.

Since the sequence ∇v_j is bounded in $L^p(\Omega')$ and $v_j \to u$ in $L^p(\Omega')$, passing to a subsequence if necessary, we may assume that $\nabla v_j \to \nabla u$ weakly in $L^p(\Omega')$. By the lower semicontinuity of the L^p–norm in the weak convergence we see that

$$\int_{\Omega'} |\nabla u|^p \, dx \leq \liminf_{j \to \infty} \int_{\Omega'} |\nabla v_j|^p \, dx \leq K \int_{\Omega'} |\nabla u_{\Omega'}|^p \, dx$$

where (8) is used in the last step. This yields (6) and the proof is complete.

There is a version of Theorem 3.2 where the assumption $u \in W_{loc}^{1,p}(\Omega)$ is not needed. To formulate the result we introduce some notation. Let w be a continuous real valued function defined on the boundary $\partial\Omega$ of a bounded open set Ω of $\mathbf{R}^n$. We let H_w^Ω denote the Perron–Wiener–Brelot solution associated with the p–harmonic equation (4) and with the boundary values w, see [HKM, Chapter 9]. Since Ω is bounded and w is continuous a unique Perron–Wiener–Brelot solution H_w^Ω with boundary values w exists, see [HKM, Theorem 9.26].

If u is a quasiminimizer, then u is locally Hölder continuous, see [CC] and [KiM]. Hence the continuity assumption in Theorem 3.3 is not an essential restriction.

Theorem 3.3 *Suppose that u is continuous in Ω. Then u is a K–quasi–minimizer in Ω if and only if for each open set $\Omega' \subset\subset \Omega$ and all disjoint open sets $\Omega_1, ..., \Omega_k \subset \Omega'$ it holds*

$$\sum_i \int_{\Omega_i} |\nabla H_u^{\Omega_i}|^p \, dx \leq K \int_{\Omega'} |\nabla H_u^{\Omega'}|^p \, dx < \infty. \tag{9}$$

Proof. The proof is similar to that of Theorem 3.2. For the sufficiency replace in the definition of the sequence v_j the functions $u_{Q_i^j}$ by the functions $H_u^{Q_i^j}$. Note that a cube is regular domain for the p–Dirichlet problem and hence the function v_j is continuous in Ω'. Now it is easy to see that the sequence $v_j, j = 1, 2, ...,$ converges locally uniformly to u in Ω' and hence no Poincaré inequality is needed.

Open problem 3.4 Is it possible to relax the continuity assumption in Theorem 3.3?

In the one dimensional case Theorem 3.2 takes a simple form.

Theorem 3.5 *Suppose that $p > 1, K \geq 1$, Δ is an open interval in $\mathbf{R}$ and $u : \Delta \to \mathbf{R}$ is a function. Then u is a K–quasiminimizer if and only if for all intervals $[a, b] \subset \Delta$ it holds*

$$\sum_{i=1}^{k} \frac{|u(x_{i+1}) - u(x_i)|^p}{(x_{i+1} - x_i)^{p-1}} \leq K \frac{|u(b) - u(a)|^p}{(b - a)^{p-1}} \tag{10}$$

whenever $a = x_1 < x_2 < ... < x_{k+1} = b$ is a partition of $[a, b]$.

Proof. Since affine functions are minimizers in the 1–dimensional case for all p, see [GG], and

$$\int_c^d |f'(t)|^p \, dt = \frac{|f(d) - f(c)|^p}{(d - c)^{p-1}} \tag{11}$$

for an affine function f, (10) follows from Theorem 3.2 for a K–quasiminimizer u.

To prove the sufficiency of (10) we first show that u is absolutely continuous in any closed interval $[a,b] \subset \Delta$. Let $(a_i, b_i), i = 1, ..., k$ be a collection of disjoint intervals in $[a,b]$. By the Hölder inequality and by (10) we obtain

$$(\sum_i |u(b_i) - u(a_i)|)^p \leq (\sum_i \frac{|u(b_i) - u(a_i)|^p}{|b_i - a_i|^{p-1}})(\sum_i |b_i - a_i|)^{p-1}$$

$$\leq K \frac{|u(b) - u(a)|^p}{(b-a)^{p-1}} (\sum_i |b_i - a_i|)^{p-1}$$

and this clearly implies absolute continuity of u on $[a,b]$.

Condition (10) also implies that $u' \in L^p_{loc}(\Delta)$. Indeed, let $[a,b] \subset \Delta$ and subdivide $[a,b]$ into intervals of equal length $< 1/i$. Approximate u on $[a,b]$ by a piecewise linear function v_i which equals u at the endpoints of subintervals. Then v_i converges uniformly to u in $[a,b]$ and it follows from (10), as in the proof of Theorem 3.2, that $v_i' \to u'$ weakly in $L^p([a,b])$, at least for a subsequence. Hence $u' \in L^p([a,b])$ and the inequality (2) follows from the lower semicontinuity of the norm with respect to the weak convergence, see the proof for Theorem 3.2. The proof follows.

Remark 3.6 The condition (10) should be compared to the condition

$$\sum_{i=1}^{j} \frac{|u(x_{i+1}) - u(x_i)|^p}{(x_{i+1} - x_i)^{p-1}} \leq M < \infty \tag{12}$$

for a function $u : [a,b] \to \mathbf{R}$. Here $x_1, x_2, ...x_{j+1}$ is any partition of $[a,b]$. Now (12) is equivalent to the fact that u is absolutely continuous on $[a,b]$ and $u' \in L^p([a,b])$, $p > 1$. The sufficiency of (12) follows as in the proof for Theorem 3.5 and the necessity is due to the Hölder inequality and the estimate

$$|u(x_{i+1}) - u(x_i)| \leq \int_{x_{i+1}}^{x_i} |u'| \, dt.$$

Inequality (10) has a reverse nature since it gives the bound for the sum in (12) in terms of the values of u at the endpoints of each interval $[a,b]$. Note that (10) is much stronger than (12). In particular (10) implies that u' is locally integrable to some exponent $q > p$ and that u is either constant or strictly monotone.

A natural domain of definition for a quasiminimizer in $\mathbf{R}$ is a closed interval $[a,b]$. Indeed, if u is a K–quasiminimizer in an open finite interval

(a,b), then u has a continuous extension to $[a,b]$ and

$$\int_c^d |u'(t)|^p\,dt \le K\frac{|u(d)-u(c)|^p}{(d-c)^{p-1}} \tag{13}$$

holds for all intervals $[c,d] \subset [a,b]$. For this and other properties of one dimensional quasiminimizers see [MS].

Open problem 3.7 It is easy to show that a non–constant (p,K)–quasiminimizer $u:\mathbf{R}\to\mathbf{R}$ is quasisymmetric in the sense of Beurling and Ahlfors. Determine the quasisymmetry constant in terms of p and K. The case $p=2$ is the most interesting.

4 Constructions for quasiminimizers

Let u be a quasiminimizer. The following theorem gives a sufficient condition for a function g so that the function $u+g$ is again a quasiminimizer.

Theorem 4.1 *Suppose that u is a K-quasiminimizer in Ω and g is a function in $W^{1,p}_{loc}(\Omega)$ such that*

$$|\nabla g| \le c|\nabla u| \tag{14}$$

a.e. in Ω where $c\in[0,K^{-1/p})$. Then the function $u+g$ is a K'– quasiminimizer with

$$K' = \frac{(1+c)^p}{(K^{-1/p}-c)^p}. \tag{15}$$

Proof. Fix an open set $\Omega' \subset\subset \Omega$ and for each function $v\in W^{1,p}_{loc}(\Omega)$ let h_v denote the minimizer of the p–Dirichlet integral with boundary values $v$in Ω', see the previous section. We need to show

$$\int_{\Omega'} |\nabla(u+g)|^p\,dx \le K'\int_{\Omega'}|\nabla h_{u+g}|^p\,dx. \tag{16}$$

The Minkowski inequality, the minimizing property of h_u and h_g, the quasiminimizing property of u and (14) yield

$$\begin{aligned}
&\|\nabla h_{u+g}\|_{L^p(\Omega')} = \|\nabla(h_{u+g}-h_g+h_g)\|_{L^p(\Omega')}\\
&\ge \|\nabla(h_{u+g}-h_g)\|_{L^p(\Omega')} - \|\nabla h_g\|_{L^p(\Omega')}\\
&\ge \|\nabla h_u\|_{L^p(\Omega')} - \|\nabla g\|_{L^p(\Omega')} \ge K^{-1/p}\|\nabla u\|_{L^p(\Omega')} - c\|\nabla u\|_{L^p(\Omega')}\\
&= (K^{-1/p}-c)\|\nabla u\|_{L^p(\Omega')}\,.
\end{aligned}$$

The Minkowski inequality, (14) and the above inequality imply

$$\| \nabla(u+g) \|_{L^p(\Omega')} \leq \| \nabla u \|_{L^p(\Omega')} + \| \nabla g \|_{L^p(\Omega')}$$

$$\leq (1+c) \| \nabla u \|_{L^p(\Omega')} \leq \frac{1+c}{K^{-1/p}-c} \| \nabla h_{u+g} \|_{L^p(\Omega')} .$$

This is the required inequality (16).

Remark 4.2 The upper bound for c in Theorem 4.1 is essentially sharp as can be easily seen from one–dimensional examples. More precisely, the function $u(t) = t$ is a minimizer, i.e. 1–quasiminimizer, in $\mathbf{R}$ for all $p > 1$. If we can now take

$$c = K^{-1/p} = 1,$$

then the function $g(t) = |t|$ satisfies $|g'(t)| \leq c|u'(t)|$ for a.e. t. However, $u+g$ is not a K'– quasiminimizer for any $K' < \infty$ because it is neither strictly monotone nor constant, see [GG]. Similar examples exist in all dimensions

Open problem 4.3 Radial quasiminimizers of power-type have been studied in the recent paper [BBM]. Very few additional constructions, besides radial functions, solutions to $\nabla \cdot A(x, \nabla u) = 0$ and Theorem 4.1, are known for quasiminimizers. It would be interesting to know other constructions. Radial quasiminimizers correspond, in some sense, one dimensional quasiminimizers in higher dimensional Euclidean spaces. It would be interesting to know if they have "extremal" properties (like, for instance, radial quasiconformal mappings).

5 Capacity estimates

Let Ω be an open and bounded subset of $\mathbf{R}^n$, $n \geq 1$ and $p > 1$. If C is a compact subset of Ω, then the pair $E = (C, \Omega)$ is called a *condenser*. The *p–capacity* of E is defined as

$$cap_p E = \inf \int_\Omega |\nabla \varphi|^p \, dx \tag{17}$$

where the infimum is taken over all functions $\varphi \in C_o^\infty(\Omega)$ such that $\varphi = 1$ on C, see Section 2. Let φ be as above. Now there is a unique function $u \in W^{1,p}(\Omega)$ such that $u - \varphi \in W_0^{1,p}(\Omega \setminus C)$ and

$$cap_p E = \int_\Omega |\nabla u|^p \, dx.$$

Moreover, the function u is p–harmonic in $\Omega \setminus C$, i.e. u is a solution of the p–harmonic equation in $\Omega \setminus C$. Set $u = 1$ on C. The function u is called the *p–potential* of C in Ω, see [HKM, Section 6.10].

Let u be the p–potential of C in Ω and $t \in (0,1)$. Set $C_t = \{x \in \Omega : u(x) \geq t\}$. Although C_t need not be a compact set in Ω we can define the p–capacity of the pair, also called a condenser, (C_t, Ω) as

$$cap_p(C_t, \Omega) = \inf \int_\Omega |\nabla v|^p \, dx \tag{18}$$

where the infimum is now taken over all functions v such that $v - u/t \in W_0^{1,p}(\Omega \setminus C_t)$ and $v = 1$ on C. Here we use the refined version of the the space $W_0^{1,p}(\Omega \setminus C_t)$ consisting of all functions $w \in W^{1,p}(\mathbf{R}^n)$ such that $w = 0$ and $w = 1$ p–quasieverywhere in the complement of Ω and in C_t, respectively. For this theory see Section 2 and [HKM, Chapter 4].

The basic equation between the p–capacities of the condensers (C_t, Ω) and E is

$$t^{p-1} cap_p(C_t, \Omega) = cap_p E. \tag{19}$$

Equation (19) becomes a double inequality

$$(\frac{\alpha}{\beta})^{p+1} t^{p-1} cap_p(C_t, \Omega) \leq cap_p E \leq (\frac{\beta}{\alpha})^{p+1} t^{p-1} cap_p(C_t, \Omega) \tag{20}$$

if, instead of a p–potential, an A–potential u of C in Ω is used. Here A refers to the degenerate second order partial differential equation

$$\nabla \cdot A(x, \nabla u) = 0 \tag{21}$$

where the operator A satisfies

$$\alpha |h|^p \leq A(x,h) \cdot h \leq \beta |h|^p, \; 0 < \alpha \leq \beta < \infty, \tag{22}$$

see [HKM, Lemma 6.19]. Equation (19) and inequality (20) are important tools in the study of boundary behavior of p– and A–harmonic functions as well as in the study of polar sets. We look for the corresponding estimates for quasiminimizers.

Since quasimininimizers do not obey the comparison principle, which is fundamental in Potential Theory, the capacity estimates of the type (20) for quasiminimizers should be based on other methods than the proof for (20). Here we develop a method which uses one dimensional quasiminimizers. For sharp estimates we need some special properties of one dimensional quasiminimizers.

A natural domain of definition for a quasiminimizer in $\mathbf{R}$ is a closed interval $[a, b]$. Indeed, if u is a K–quasiminimizer in an open interval (a, b), then u has a continuous extension to $[a, b]$ and

$$\int_c^d |u'(t)|^p \, dt \leq K \frac{|u(d) - u(c)|^p}{(d-c)^{p-1}} \tag{23}$$

holds for all intervals $[c, d] \subset [a, b]$. For this and other properties of one dimensional quasiminimizers see Section 3 and [MS].

We say that a K–quasiminimizer $u : [0, 1] \to [0, 1]$ is a *normalized* K–quasiminimizer if $u(0) = 0$ and $u(1) = 1$. The following lemma is immediate.

Lemma 5.1 *Suppose that u is a normalized (p, K)–quasiminimizer. Then*

$$u(t) \leq K^{1/p} t^{(p-1)/p}$$

for each $t \in [0, 1]$.

Proof. By the Hölder inequality

$$u(t) = \int_0^t u'(s) \, ds \leq t^{(p-1)/p} (\int_0^1 u'(s)^p \, ds)^{1/p}$$

$$\leq t^{(p-1)/p} [K \frac{(u(1) - u(0))^p}{(1-0)^{p-1}}]^{1/p} = K^{1/p} t^{(p-1)/p}$$

as required.

Next we review some results from [MS] which will be needed in the sequel. We use the same notation as in [MS] where the higher regularity properties of one dimensional quasiminimizers were considered in detail. From [MS, Theorem 4] it follows that there is a function $p_1 : (1, \infty) \times [1, \infty) \to (1, \infty]$ and for each triple $(p, K, s) \in (1, \infty) \times [1, \infty) \times (1, p_1(p, K^{1/p}))$ a number $K_1 = K_1(p, K, s)$ such that if u is a normalized (p, K)–quasiminimizer, then u is also a (s, K_1)–quasiminimizer. The function p_1 satisfies for each $p \in (1, \infty)$and $K \geq 1$:

$$p_1(p, K) > p \tag{24}$$

$$\lim_{K \to 1} p_1(p, K) = \infty = p_1(p, 1) \tag{25}$$

$$\lim_{K \to \infty} p_1(p, K) = p \tag{26}$$

and the number K_1 has the property $K_1(p, 1, s) = 1$ for each $p, s > 1$.

From the above property we obtain an improved version of Lemma 5.1:

Corollary 5.2 *Suppose that u is a normalized (p, K)–quasiminimizer. Then for each $s \in (1, p_1(p, K^{1/p}))$*

$$u(t) \leq K_1(p, K, s)^{1/s} t^{(s-1)/s} \tag{27}$$

for every $t \in [0, 1]$.

Remark 5.3 The only normalized 1–quasiminimizer is $u(t) = t$. Note that (27) reduces to $u(t) \leq t$ for $K = 1$.

Suppose that u is a normalized (p, K)–quasiminimizer. Then u is strictly increasing and continuous. By [MS, Theorem 14] there is a function $p_2 : (1, \infty) \times [1, \infty) \to (1, \infty]$ and for each $q \in p_2(p/(p-1), K^{1/(p-1)})$ a number $K_2 = K_2(p, K, q)$ such that the inverse function v of u is for each $q \in (1, p_2(p/(p-1), K^{1/(p-1)}))$ also a (q, K_2)–quasiminimizer. The function p_2 and the number $K_2 = K_2(p, K, q)$ satisfy

$$\begin{aligned} \lim_{K \to 1} p_2(p/(p-1), K) &= \infty = p_2(p/(p-1), 1) && (28) \\ \lim_{K \to \infty} p_2(p/(p-1), K) &= 1 && (29) \\ K_2(p, 1, q) &= 1. && (30) \end{aligned}$$

Remark 5.4 For $p = 2$ the functions $p_1(2, K)$ and $p_2(2, K)$ have explicit expressions, see [MS],

$$p_1(2, \sqrt{K}) = 1 + \sqrt{K/(K-1)}$$

$$p_2(2, K) = \sqrt{K/(K-1)}$$

for $K > 1$. The aforementioned functions p_1, p_2 and the numbers K_1, K_2 can be numerically computed for all argument values, see [D'AS] and [MS]. Moreover, all these results are sharp. Note the open ended property for the exponents s and q.

We state the counterpart to (18) as two separate theorems although their proofs follow from the same principle. We let the functions p_1, p_2 and the numbers K_1, K_2 be as above.

Suppose that $E = (C, \Omega)$ be a condenser in $\mathbf{R}^n, n \geq 1$, where Ω is a bounded open set. Let u be a (p, K)– quasiminimizer in $\Omega \setminus C$ with boundary values 0 on $\partial\Omega$, 1 on C, i.e. $u - \varphi \in W_0^{1,p}(\Omega \setminus C)$ where $\varphi \in C_0^\infty(\Omega)$ and $\varphi = 1$ on C. Write $C_t = \{x \in \Omega : u(x) \geq t\}$.

Theorem 5.5 *For each $s \in (1, p_1(p/(p-1), K^{1/p})$ there is a number $\kappa_1 = \kappa_1(p, K, s) < \infty$ such that*

$$cap_p E \leq \kappa_1 t^{p-s/(s-1)}\, cap_p(C_t, \Omega). \tag{31}$$

The number κ_1 has the following property for each $p > 1$ and $s \in (1, p_1(p/(p-1), K^{1/p(p-1)})$:

$$\lim_{K \to 1} \kappa_1(p, K, s) = 1 = \kappa_1(p, 1, s)$$

Remark 5.6 For $K = 1$ inequality (31) and the properties of p_1 give $cap_p E \leq t^{p-1}\, cap_p(C_t, \Omega)$. Inequality (31) does not reduce to the right hand side of (20) if the A–potential u of C in Ω is considered as a quasiminimizer because now an A–potential of C in Ω is a $(\beta/\alpha)^p$–quasiminimizer in $\Omega \setminus C$. Since $p_1(p/(p-1), K^{1/p}) > p/(p-1)$, the exponent s can be chosen $> p/(p-1)$.

Proof for Theorem 5.5. Set $u = 1$ on C. Then $u \in W^{1,p}(\Omega)$ and write

$$\Omega_t = \{x \in \Omega : u(x) > t\},\ 0 \leq t < 1.$$

Although Ω_t need not be an open subset of Ω, for each $0 \leq t < t' \leq 1$ we can define a condenser $(C_{t'}, \Omega_t)$ and its p–capacity $cap_p(C_{t'}, \Omega_t)$ as before using the refined Sobolev functions. Note that $\Omega_0 = \Omega$ provided that C is a set of positive p–capacity and Ω is a domain. For each $0 \leq t < t' \leq 1$ the quasiminimizing property of u yields

$$cap_p(C_{t'}, \Omega_t) \leq (t' - t)^{-p} \int_{\Omega_t \setminus C_{t'}} |\nabla u|^p\, dx \leq K cap_p(C_{t'}, \Omega_t). \tag{32}$$

Fix an interval $[a, b] \subset [0, 1]$ and let $a = t_0 < t_1 < ... < t_k = b$ be a partition of $[a, b]$. Next we employ the well known separation inequality for capacities of condensers, see [HKM, Theorem 2.6]: For the condensers $(C_{t_i}, \Omega_{t_{i-1}})$, $i = 1, 2, ..., k$, this gives

$$\sum_{i=1}^{k} cap_p(C_{t_i}, \Omega_{t_{i-1}})^{-1/(p-1)} \leq cap_p(C_{t_k}, \Omega_{t_0})^{-1/(p-1)} \tag{33}$$

because

$$C_{t_k} \subset \Omega_{t_{k-1}} \subset C_{t_{k-1}} \subset \Omega_{t_{k-2}} \subset\ ...\ \subset \Omega_{t_0}.$$

Set

$$\varphi(t) = \int_{\{u(x) < t\}} |\nabla u|^p\, dx.$$

Then $\varphi : [0, 1] \to [0, \beta]$,

$$\beta = \int_{\Omega \setminus C} |\nabla u|^p\, dx,$$

is a continuous strictly increasing function with $\varphi(0) = 0$ and $\varphi(1) = \beta$. Now (32) and (33) yield

$$\sum_{i=1}^{k}(t_i - t_{i-1})^{p/(p-1)}(\varphi(t_i) - \varphi(t_{i-1}))^{1/(1-p)}$$
$$\leq K^{1/(p-1)}(b-a)^{p/(p-1)}(\varphi(b) - \varphi(a))^{1/(1-p)}. \tag{34}$$

Note that $\nabla u = 0$ almost everywhere on the set where $u =$const.

Let $\psi : [0, \beta] \to [0, 1]$ denote the inverse function of φ. Writing (34) for the inverse function ψ we obtain

$$\sum_{i=1}^{k}(\psi(t'_i) - \psi(t'_{i-1}))^{p/(p-1)}(t'_i - t'_{i-1})^{1/(1-p)}$$
$$\leq K^{1/(p-1)}(\psi(b') - \psi(a'))^{p/(p-1)}(b' - a')^{1/(1-p)} \tag{35}$$

where $\varphi(a) = a'$, $\varphi(b) = b'$ and $\varphi(t_i) = t'_i, i = 0, 1, \ldots, k$. Thus (35) holds for an arbitrary partition

$$a' = t'_0 < t'_1 < \ldots < t'_k = b'$$

of the interval $[a', b'] \subset [0, \beta]$. By Theorem 3.5 the function ψ is a $(p/(p-1), K^{1/(p-1)})$–quasiminimizer. Since $t \to \psi(\beta t)$ is a normalized quasiminimizer, Corollary 5.2 yields for each $s \in (1, p_1(p/(p-1), K^{1/p(p-1)}))$

$$\psi(\beta t) \leq c_1^{1/s} t^{(s-1)/s} \tag{36}$$

where $c_1 = c_1(p, K, s)$. For the inverse function φ of ψ this means that

$$\varphi(t) \geq \beta c_1^{1/(1-s)} t^{s/(s-1)},\ t \in [0, 1]. \tag{37}$$

By the quasiminimizing property of u

$$cap_p(C_t, \Omega) \geq K^{-1} \int_{\{u<t\}} |\nabla(\frac{u}{t})|^p\, dx = K^{-1} t^{-p} \varphi(t)$$

and hence we obtain from (37)

$$cap_p(C, \Omega) \leq \beta \leq c_1^{1/(s-1)} t^{-s/(s-1)} \varphi(t)$$
$$\leq K c_1^{1/(s-1)} t^{p-s/(s-1)} cap_p(C_t, \Omega).$$

It is easy to check that the number

$$\kappa_1 = \kappa_1(p, K, s) = K c_1^{1/(1-s)}$$

has the required property. The proof follows.

The next theorem gives the counterpart of the left hand side of (20) for quasiminimizers.

Theorem 5.7 *For each* $q \in (1, p_2(p,K))$ *there is* $\kappa_2 = \kappa_2(p,K,q)$ *such that*

$$\frac{t^{p-(q-1)/q}}{\kappa_2}\, cap_p(C_t,\Omega) \le cap_p E. \tag{38}$$

The number κ_2 *satisfies for each* $p > 1$ *and* $q \in (1, p_2(p,K))$*:*

$$\lim_{K\to 1} \kappa_2(p,K,q) = 1 = \kappa_2(p,1,q).$$

Proof. We proceed as in the proof of Theorem 5.5. Since the function φ is the inverse function of ψ and ψ is a $(p/(p-1), K^{1/(p-1)})$–quasiminimizer, for each $q \in (1, p_2(p,K))$, see Section 3, there is a number $K_2 = K_2(p/(p-1), K^{1/(p-1)}, q)$ such that the function φ is a (q, K_2)–quasiminimizer in $[0,1]$. Since φ/β is a normalized quasiminimizer, Corollary 5.2 yields

$$\varphi(t) \le K_2^{1/q} t^{(q-1)/q}\beta,\ t \in [0,1]$$

and since

$$cap_p(C_t,\Omega) \le t^{-p}\varphi(t)$$

we obtain from the quasiminimizing property of u that

$$cap_p(C,\Omega) \ge \frac{\beta}{K} \ge \frac{t^{p-(q-1)/q}}{KK_2^{1/q}} cap_p(C_t,\Omega)$$

$$= \frac{t^{p-(q-1)/q}}{\kappa_2} cap_p(C_t,\Omega)$$

where the number $\kappa_2 = KK_2^{1/q}$ has the required property. The proof follows.

Remark 5.8 Remark 5.6 also applies to Theorem 5.7: For $K = 1$ inequality (38) reduces to $cap_p E \ge t^{p-1}\, cap_p(C_t,\Omega)$. However, the exponents and the constants on the right and left hand side of (31) and (38) are different.

Open problem 5.9 Are the estimates (31) and (38) sharp for $n \ge 1$? As J. Björn pointed out the Hölder continuity result (27) in Corollary 5.2 need not be sharp although the higher integrability exponent result for the derivative of a one dimensional quasiminimizer is sharp.

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