Bernoulli Free-boundary Problems

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Linköping Lectures 10 – 14 August 2009

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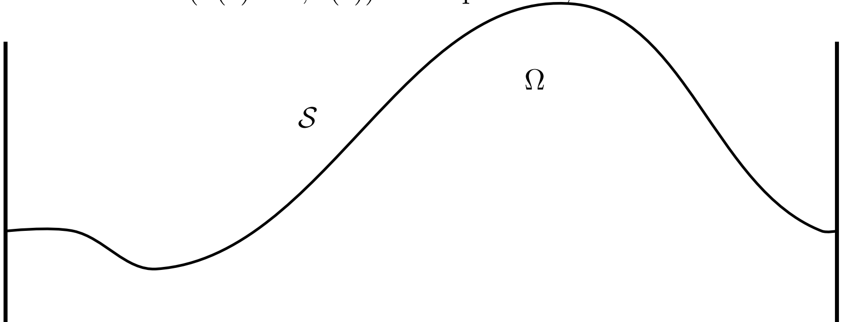
Lecture 1: Bernoulli Free Boundaries

Let Ω be the domain below $\mathcal{S}$ in the (X,Y)-plane where

$$\begin{aligned}
&\mathcal{S} := \{(u(s), v(s)) \; : \; s \in \mathbb{R}\} \text{ is } 2\pi\text{-periodic,}\\
&(u,v) \text{ is injective and absolutely continuous,}\\
&u'(s)^2 + v'(s)^2 > 0 \text{ for almost all } s,\\
&s \mapsto (u(s) - s, v(s)) \text{ is } 2\pi\text{–periodic,}
\end{aligned}$$



Dirichlet Problem

Consider the problem of finding ψ with

$$\begin{aligned}
&\psi \in C(\overline{\Omega}) \cap C^2(\Omega),\\
&\Delta\psi = 0 \text{ in } \Omega,\\
&\psi \text{ is } 2\pi\text{–periodic in } X,\\
&\nabla\psi(X,Y) \to (0,1) \text{ as } Y \to -\infty \ \text{ uniformly in } X,\\
&\psi \equiv 0 \text{ on } \mathcal{S}.
\end{aligned}$$

By classical theory a solution always exists and, by the Maximum Principle, $\psi < 0$ and $\nabla\psi$ is nowhere zero on Ω.

Bernoulli Free Boundary Problems

A Bernoulli free-boundary problem is one of determining those curves $\mathcal{S}$ with the property that the solution of this Dirichlet problem satisfies an additional inhomogeneous Neumann condition

$$\boxed{\frac{\partial \psi}{\partial n} = h(Y) \text{ almost everywhere on } \mathcal{S}}$$

where h is given and n denotes the outward normal to Ω at points of $\mathcal{S}$.

As the outward normal derivative of ψ on $\mathcal{S}$ is non-negative, by the maximum principle, $h \geq 0$ on $\mathcal{S}$ is necessary for the existence of solutions. Because formally the tangential derivative of ψ is zero almost everywhere it is convenient to reformulate the Neumann condition as

$$\text{for every } (X, Y) \in \mathcal{S},$$
$$|\nabla\psi(X_1, Y_1)|^2 \to \lambda(Y) \text{ as } (X_1, Y_1) \to (X, Y) \text{ in } \Omega$$

where $\lambda = h^2$. We will consider only functions λ which are continuous on $\mathcal{R}(v)$where $\mathcal{R}(v)$ is the range of v, a compact interval, and that λ is real-analytic on the open set of full measure where it is non-zero. The real-analyticity hypothesis is made for technical convenience and our results have analogues for other classes of λ.
Since $\lambda(Y)$ is continuous on $\mathcal{S}$, it follows that $|\nabla\psi|$ is continuous on $\overline{\Omega}$.

Special case: Stokes waves

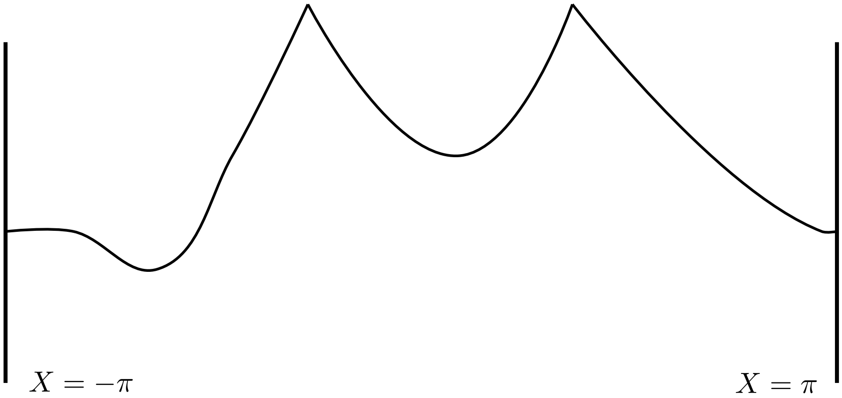
- $\lambda(Y) = 1 - 2gY$ where $g > 0$ is the acceleration due to gravity
- ψ is the stream function
- $(\psi_Y, -\psi_X)$ is the steady velocity field
- the Dirichlet and Neumann boundary condition mean that $\mathcal{S}$ is a streamline at which the pressure in the flow is a constant
- a point on $\mathcal{S}$ where the velocity is zero is called a stagnation point

Although λ is affine in the case of Stokes waves, there is nothing special about the the theory of Stokes waves that distinguishes it from the general theory.

Stagnation Points

$(X_0, Y_0) \in \mathcal{S}$ is a stagnation point if $\lambda(Y_0) = 0$

Only at stagnation points can $\mathcal{S}$ not be smooth, and there is a corner at each stagnation point.



Notation

- $L^p_{2\pi}$, $p \geq 1$, denotes the space of 2π-periodic locally p^{th}-power summable real-valued functions.
- For $p \geq 1$, let $W^{1,p}_{2\pi}$ be the space of functions $w \in L^p_{2\pi}$ with weak first derivatives $w' \in L^p_{2\pi}$

Conjugation Operator or Hilbert Transform $\mathcal{C}u$ is defined almost everywhere for any 2π-periodic locally integrable functions u by the Cauchy Principal Value integral

$$\mathcal{C}u(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(y) \cot(\tfrac{1}{2}(x-y))dy.$$

Alternatively,

$$\mathcal{C} \sin kx = -\cos kx, \quad \mathcal{C} \cos kx = \sin kx, \quad k \in \mathbb{N}, \quad \mathcal{C}1 = 0,$$

defines $\mathcal{C}$ for square-integrable functions.

- $\mathcal{C}$ is a bounded linear operator on $L^p_{2\pi}$, $1 < p < \infty$ but not in $L^1_{2\pi}$ or $L^\infty_{2\pi}$.
- $\mathcal{H}^{1,1}_{\mathbb{R}}$ be the real *Hardy space* of functions $w \in W^{1,1}_{2\pi}$ with w' in the usual *Hardy space* $\mathcal{H}^1_{\mathbb{R}} := \{u \in L^1_{2\pi} : \mathcal{C}u \in L^1_{2\pi}\}$.
- $\mathcal{H}^{1,1}_{\mathbb{R}}$ is a Banach algebra and $\lambda(u) \in \mathcal{H}^{1,1}_{\mathbb{R}}$ when $u \in \mathcal{H}^{1,1}_{\mathbb{R}}$, if λ is Lipschitz continuous.
- Let $\mathcal{H}^\infty_{\mathbb{R}}$ denote the real Hardy spaces of 2π-periodic functions u such that $u, \mathcal{C}u \in L^\infty_{2\pi}$ and let $\mathcal{H}^{1,\infty}_{\mathbb{R}}$ be the space of absolutely continuous functions with $w' \in \mathcal{H}^\infty_{\mathbb{R}}$.
- The k-times continuously differentiable functions on an interval I are denoted by $C^k(I)$.
- Hölder continuous functions are denoted by $C^{k,\alpha}(I)$.

Complex Hardy Spaces.

Let $D \subset \mathbb{C}$ denote the open unit disc. For a holomorphic function $f : D \to \mathbb{C}$, let $f_r(t) = f(re^{it})$ for $t \in \mathbb{R}$ and $r \in (0,1)$.

The *Nevanlinna class* N consists of complex analytic functions $f : D \to \mathbb{C}$ such that

$$\sup_{r\in(0,1)} \int_0^{2\pi} \log^+ |f(re^{it})| dt < \infty.$$

If $f \in N$, $\lim_{r \nearrow 1} f(re^{it})$, denoted by $f^*(t)$, $t \in \mathbb{R}$, exists almost everywhere and $\log|f^*| \in L^1_{2\pi}$ if $f \not\equiv 0$.

A function $f \in N$ belongs to the *Nevanlinna–Smirnov class* N^+ if

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{it})| dt = \int_0^{2\pi} \log^+ |f^*(t)| dt, \quad \log^+ = \max\{0, \log\} \tag{1}$$

- It is well known that, for any $p \in (0, \infty]$,

$$||f||_p := \lim_{r \to 1} \|f_r\|_{L^p_{2\pi}} = \sup_{r \in (0,1)} \|f_r\|_{L^p_{2\pi}} \text{ is well defined}$$

- The Hardy class $\mathcal{H}^p_{\mathbb{C}}$ is the set of f with $||f||_p < \infty$.
- Note that $\mathcal{H}^p_{\mathbb{C}} \subset N^+$ and, for $f \in \mathcal{H}^p_{\mathbb{C}}$, $f^* \in L^p_{2\pi}$, $\|f^*\|_{L^p_{2\pi}} = ||f||_p$ and $\log |f^*| \in L^1_{2\pi}$ if $f \not\equiv 0$.
- By a theorem of *Smirnov*, $F \in N^+$ and $F^* \in L^p_{2\pi}$ together imply that $F \in \mathcal{H}^p_{\mathbb{C}}$.
- Moreover $u \in \mathcal{H}^1_{\mathbb{R}}$ if and only if $u + i\mathcal{C}u = U^*$ for some $U \in \mathcal{H}^1_{\mathbb{C}}$.

Equation (**A**)

Theorem

Let u, v, ψ *be a solution of the Bernoulli problem and let* φ *be a harmonic conjugate of* $-\psi$. *Then* $\varphi + i\psi$ *is an injective conformal mapping of* Ω *onto the open lower half-plane which has an extension as a homeomorphism from* $\overline{\Omega}$ *onto the closed lower half-plane. Let* Z *be its inverse. Then* $t \mapsto Z(iy - t)$ *is an absolutely continuous parametrization of the streamline* $\{(X, Y) : \psi(X, Y) = y\}$ *for all* $y \leq 0$. *Let*

$$w(t) = \text{Im } Z(-t), \quad t \in \mathbb{R}.$$

Then $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *satisfies*

$$\boxed{\lambda(w)\{w'^2 + (1 + \mathcal{C}w')^2\} = 1.} \qquad (\mathbf{A})$$

Moreover $1/W \in N^+$ *where* $W \in \mathcal{H}^1_{\mathbb{C}}$ *is such that* $W^* = w' + i(1 + \mathcal{C}w')$.

Equation (B)

The extent to which this equation is equivalent to

$$\boxed{\lambda(w)\big(1+\mathcal{C}w'\big)+\mathcal{C}\big(\lambda(w)w'\big)=1,} \qquad (\mathbf{B})$$

is less obvious.

Equation (B) is the Euler-Lagrange equation of the functional

$$\boxed{\mathcal{J}(w)=\int_{-\pi}^{\pi}\big\{\Lambda(w)\big(1+\mathcal{C}w'\big)-w\big\}\,dt, \quad w\in\mathcal{H}^{1,1}_{\mathbb{R}},}$$

which has a natural physical interpretation, when Λ is a primitive of λ

Theorem

For $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ let $W \in \mathcal{H}^{1}_{\mathbb{C}}$ be such that $W^* = w' + i(1+\mathcal{C}w')$. Then the following are equivalent.

(i) *w satisfies* $(\mathbf{B})$ *and* $\lambda(w) \geq 0$;

(ii) *w satisfies* $(\mathbf{A})$ *and* $1/W \in N^+$.

The Key

Riemann-Hilbert Theory

Euler-Lagrange equation (**B**) $\quad \lambda(w)\big(1+\mathcal{C}w'\big)+\mathcal{C}\big(\lambda(w)w'\big)=1$ can be written in complex form as

$$\lambda(w)w'+i(-1+\mathcal{C}(\lambda(w)w'))=\lambda(w)w'-i\lambda(w)(1+\mathcal{C}w')$$

In other words, if

$$V^*=\lambda(w)w'+i(-1+\mathcal{C}(\lambda(w)w')) \text{ and } W^*=w'+i(1+\mathcal{C}w')$$

then

$$V^*=\lambda(w)\overline{W}^*$$

Therefore $V^*W^*=\lambda(w)|W^*|^2$. Hence, since VW is the Poisson's integral of its boundary data, and the Poisson kernel is smooth, then VW is a real-valued holomorphic function. Hence, by Cauchy-Riemann equations it is a constant. Hence

$$\lambda(w)\{{w'}^2+(1+\mathcal{C}w')^2\}=\lambda(w)|W^*|^2=\text{const}$$

which gives (**A**)

Caveat

Let $V = W : D \to \mathbb{C}$ be the holomorphic functions defined by

$$V(z) = W(z) = \frac{1 - iz}{z - i} = \left(\frac{1 - iz}{z - i}\right)\left(\frac{\overline{z} + i}{\overline{z} + i}\right)$$
$$= \frac{\overline{z} + z + i(1 - |z|^2)}{|z - i|^2}$$

Thus $V^* = \overline{V}^*$ but VW is not a constant. The key result is

Theorem

Suppose $V, W \in \mathcal{H}^1_{\mathbb{C}}$ and $V^* = a\overline{W}^*$ where a is a bounded measurable function on ∂D. Then the following are equivalent.

- $a \geq 0$
- $a|W^*|^2$ *is integrable*
- $a|W^*|^2 \equiv \text{const}$

More about **(B)**

$$\lambda(w)\big(1+\mathcal{C}w'\big)+\mathcal{C}\big(\lambda(w)w'\big)=1$$

is the Euler-Lagrange equation of

$$\mathcal{J}(w)=\int_{-\pi}^{\pi}\big\{\Lambda(w)\big(1+\mathcal{C}w'\big)-w\big\}\,dt,\quad w\in\mathcal{H}^{1,1}_{\mathbb{R}}.$$

Also it can be rewritten

$$\begin{aligned} 2\lambda(w)\mathcal{C}w' &= 1-\lambda(w)+\boxed{\lambda(w)\mathcal{C}w'-\mathcal{C}(\lambda(w)w')} \\ &= 1-\lambda(w)+\boxed{\mathcal{F}(w)} \end{aligned}$$

Note that $w\mapsto\mathcal{C}w'$ is first order and self-adjoint, with Fourier multipliers $|k|,\ k\in\mathbb{Z}$.

Moreover $\mathcal{F}$ is a smoothing operator, in fact:

Theorem

Suppose that Λ is convex and $\lambda = \Lambda'$. Then $\mathcal{F}(u)(x) \geq 0$ almost everywhere. (Moreover, $\mathcal{F}$ is smoothing)

Proof. Let

$$G(u)(x,y) = \Lambda(u(y)) - \Lambda(u(x)) - \lambda(u(x))(u(y) - u(x))$$

$G \geq 0$ because Λ is convex. Therefore,

$$\lambda(u(x))\mathcal{C}u'(x) - \mathcal{C}(\lambda(u)u')(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\lambda(u(x)) - \lambda(u(y)))u'(y)}{\tan((x-y)/2)}dy$$

$$= \frac{-1}{2\pi}\int_{-\pi}^{\pi} \frac{(\partial/\partial y)G(u)(x,y)}{\tan((x-y)/2)}dy = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{G(u)(x,y)}{\sin^2((x-y)/2)}dy \geq 0.$$

$\square$

Smoothing

Suppose λ is smooth. Then

- $u \in W^{1,2}_{2\pi} \Rightarrow \mathcal{F}(u) \in L^{\infty}_{2\pi}$ and sequentially continuous from the weak $W^{1,2}2\pi$-topology, into $L^p_{2\pi}$, $1 \le p < \infty$, with the strong L_p-topology.
- If $u \in W^{1,p}_{2\pi}$ for $2 < p < \infty$. Then $\mathcal{F}(u) \in C^{1-\frac{2}{p}}$.
- $u \in C1^{\alpha}$, $\alpha \in (0,1) \Rightarrow \mathcal{F}(u) \in C^{1,\delta}$, $0 < \delta < \alpha$.

So if $\lambda \neq 0$, a bootstrap gives $u \in C^{\infty}$. Then an independent argument gives that u is real-analytic because λ is real-analytic.

Equation (B) and Bernoulli free boundaries

Theorem

We have observed that every Bernoulli free boundary gives a solution w of

$$\lambda(w)\{w'^2 + (1+\mathcal{C}w')^2\} = 1 \qquad \text{((A))}$$

In addition it follows that

$$\left.\begin{array}{c} w \ \textit{satisfies}\ (\mathbf{B}); \\ \lambda(w) \geq 0; \\ t \mapsto (-(t+\mathcal{C}w(t)), w(t)) \ \ \textit{injective on}\ \mathbb{R}. \end{array}\right\} \qquad (\mathbf{C})$$

Conversely, suppose that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies $(\mathbf{C})$. *Let*

$$\mathcal{S} = \{(-(t+\mathcal{C}w(t)), w(t)) : \ t \in \mathbb{R}\}$$

and let Ω be the open domain below $\mathcal{S}$. There exists a conformal mapping ω of Ω onto $\mathbb{C}^-$ such that $\mathcal{S}$ gives a solution of a Bernoulli free boundary problem.

Correspondence

There is a one-to-one correspondence between solutions of Bernoulli free boundary problems with $|\nabla \psi|$ bounded and solutions $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ of $(\mathbf{C})$

The question is: can we say when a solution of $(\mathbf{B})$ satisfies $(\mathbf{C})$

t_0 is called a stagnation point when $\lambda(w(t_0)) = 0$, and solutions with stagnation points are called singular. The set $\mathcal{N}(w)$ of stagnation points is closed.

If

$$\lambda \geq 0, \quad \log \lambda \text{ is non-constant, concave, and}$$
$$\lambda' \leq 0 \text{ where } \lambda \neq 0 \text{ on } \mathcal{R}(w),$$

a solution of $(\mathbf{B})$ defines a *non-self-intersecting* curve $\mathcal{S}$ and $\mathcal{S}$ is a Bernoulli free boundary *provided w has at most countably many stagnation points.*

Duality

Recall equation $(\mathbf{B})$ in the form

$$\lambda(w)w' + i(-1 + \mathcal{C}(\lambda(w)w')) = \lambda(w)\big(w' - i(1 + \mathcal{C}w')\big)$$

which can be re-written

$$\begin{aligned} -(w' + i(1 + \mathcal{C}w')) &= \frac{1}{\lambda(w)}\Big(-\lambda(w)w' - i(1 + \mathcal{C}(-\lambda(w)w')\Big) \\ &= \frac{1}{\lambda(w)}\Big(v' - i(1 + \mathcal{C}(v')\Big) \end{aligned}$$

where $v = -\lambda(w)$. Suppose that $\lambda(w) \geq 0$ so that $(\mathbf{A})$ holds also.

Let $\widetilde{w}(t) = -\int_0^t \lambda(w(x))w'(x)dx$ and $\lambda(w(t))\widetilde{\lambda}(\widetilde{w}(t)) \equiv 1$

Then $\widetilde{w}(t) = -\int_0^t \lambda(w(x))w'(x)dx$ is a solution of $(\mathbf{A})$ and $(\mathbf{B})$ with $\widetilde{\lambda}$ instead of λ

Dual Stokes Waves

The Stokes wave free boundary conditions are that the harmonic stream function satisfy

$$\boxed{\psi \equiv 0 \text{ and } |\nabla \psi|^2 + 2gy \equiv 1 \text{ on } \mathcal{S}}$$

The dual problem corresponds to to a free-boundary problem for the "dual stream function" $\widetilde{\psi}$:

$$\boxed{\widetilde{\psi} \equiv 0, \quad (4gy+1)|\nabla \widetilde{\psi}|^4 \equiv 1}$$

at the "dual" free boundary $\widetilde{\mathcal{S}}$

These two apparently distinct Bernoulli problems are equivalent

Self-Duality

An example

It is natural to ask if there are λs such that $\widetilde{\lambda} \equiv \lambda$.
Consider the case $\lambda \in C(\mathbb{R})$, $\lambda(v) > 0$, $\forall v \in \mathbb{R}$.

Theorem

(i) Suppose $f : [0, +\infty) \to [0, +\infty)$ is continuously differentiable, $f(0) = 0$, $f' > 0$, and $f'(0) = 1$. Let

$$\Lambda(w) = \begin{cases} f(w), & \text{if } w \geq 0, \\ -f^{-1}(-w), & \text{if } w < 0. \end{cases} \tag{2}$$

Then $\lambda = \Lambda'$ is self-dual.

(ii) Conversely, if λ is self-dual, then

$$\Lambda(w) := \int_0^w \lambda(v) dv, \quad w \in \mathbb{R}$$

has the form (2).

Regularity of Solutions of $(\mathbf{B})$

Without hypotheses on sign of $\lambda(w)$ we observe how $\lambda(w) \neq 0$ relates to the regularity of solutions w of $(\mathbf{B})$.

Theorem

When $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *is a solution of* $(\mathbf{B})$

- $\log|\lambda(w)| \in L^1_{2\pi}$
- $\lambda(w) > 0$ *on a set of positive measure*
- w *is real-analytic on the open set of full measure* $\lambda(w) \neq 0$

As a corollary, if $\mathcal{S}$, ψ *is a Bernoulli free boundary, then*

- $\mathcal{S}$ *and* ψ *are real-analytic in a neighbourhood of any point of* $\mathcal{S}$ *that is not a stagnation point,*
- $\nabla\psi$ *is continuous in the closure of* Ω

How zeros of λ affects the smoothness of w

Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of $(\mathbf{B})$. Suppose that $\varrho > 0$ is such that for all $x_0 \in \mathcal{R}(w)$ with $\lambda(x_0) = 0$,

$$|\lambda(x)| \leq \text{constant } |x - x_0|^\varrho \text{ for all } x \in \mathcal{R}(w).$$

Let

$$p(\varrho) = \frac{\varrho + 2}{\varrho} \quad \text{and} \quad r(\varrho) = \frac{\varrho + 2}{\varrho + 1}.$$

(a) The following are equivalent:

(i) $w \in W^{1,p(\varrho)}_{2\pi}$ ($w \in W^{1,3}_{2\pi}$ if λ is Lipschitz);
(ii) w is real-analytic on $\mathbb{R}$;
(iii) $\lambda(w) > 0$ on $\mathbb{R}$.

(b) The function w is real-analytic if

$$\lambda(w) \geq 0 \text{ and } -(1+\mathcal{C}w') + iw' = \big| -(1+\mathcal{C}w') + iw' \big| \, e^{i\vartheta},$$

where $\vartheta = \vartheta_1 + \vartheta_2$ with ϑ_1 continuous and $\|\vartheta_2\|_\infty < \pi/(2p(\varrho))$.

($\|\vartheta_2\|_\infty < \pi/6$ if λ is Lipschitz)

(c) If $w \in W^{1,r(\varrho)}_{2\pi}$ then $\lambda(w) \geq 0$ ($w \in W^{1,3/2}_{2\pi}$ if λ is Lipschitz)

(d) If $\varrho = 0$, which amounts to no additional hypothesis since λ is continuous and $\mathcal{R}(w)$ is compact, then $\lambda(w) \geq 0$ if $w \in W^{1,2}_{2\pi}$.

It is not known whether there are solutions of (**B**) which do not satisfy (**A**) for which $\lambda(w)$ changes sign.

There are however solutions w of (**A**) and (**B**) for which $\lambda(w)$ has zeros - the famous Stokes waves

Dimension of the Set of Stagnation Points

It follows from Theorem 7 that $\mathcal{N}(w)$ has measure 0. The following result implies that its dimension is not greater than 2/3 if λ is Lipschitz continuous. Note that the lower Minkowski dimension $\dim_M$, bounds the Hausdorff dimension from above.

Theorem

Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of $(\mathbf{A})$ and $(\mathbf{B})$ where λ is such that

$$c|x-x_0|^{\varrho} \leq \lambda(x) \leq C|x-x_0|^{\varrho}, \quad c,\, C,\, \varrho > 0,$$

for all x in a neighbourhood of x_0 in $\mathcal{R}(w)$ when $\lambda(x_0) = 0$. Let $q(\varrho) = (\varrho+2)/2$. Then

$$\dim_M \mathcal{N}(w) \leq 1/q(\varrho).$$

If $w \in W^{1,p}_{2\pi}$, $p > 1$, then

$$\dim_M \mathcal{N}(w) \leq 1-(p/p(\varrho)), \;\; 1 < p < p(\varrho), \;\; \mathcal{N}(w) = \emptyset \textit{ when } p \geq p(\varrho).$$

Jordan Curves

We would like to use the functional $\mathcal{J}$ and its Euler-Lagrange equation (**B**), without further qualification to study Bernoulli free-boundary problems. Suppose

$$\lambda \geq 0, \quad \log \lambda \text{ is non-constant, concave, and}$$
$$\lambda' \leq 0 \text{ where } \lambda \neq 0 \text{ on } \mathcal{R}(w)$$

and w, a solution of (**B**), has at most countably many stagnation points. Let

$$\vartheta = \mathcal{C}\big(\log \sqrt{\lambda(w)}\big)$$

Theorem

If $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *satisfies* (**A**) *and* (**B**), *then*

$$\sqrt{\lambda(w)}\, w' = \sin\vartheta \quad \textit{and} \quad \sqrt{\lambda(w)}(1 + \mathcal{C}w') = \cos\vartheta \tag{3}$$

and $\vartheta(t) \in \big(-\pi/2, \pi/2\big)$. *Hence* $1 + \mathcal{C}w' > 0$, *almost everywhere. For smooth functions* $1 + \mathcal{C}w' > 0$ *everywhere.*

Important Open Question

The hypotheses of this theorem on λ are valid when $\lambda(w) = 1 - 2gw$ for any $g > 0$.

Unfortunately even in that case it is not known whether the requirement that $\mathcal{N}(w)$ be denumerable is necessary.

In fact no examples are known in which $\mathcal{N}(w) \cap [0, 2\pi)$ contains more than one point when $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies $(\mathbf{A})$ and $(\mathbf{B})$.

Can a solution w of $(\mathbf{A})$ and $(\mathbf{B})$ have uncountably many stagnation points?