Title: The Continuing Saga of the Infinity Laplace Equation

Abstract: Gunnar Aronsson invented the "infinity-Laplace" equation $\Delta_\infty u = \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j} = 0$ over 40 years ago. Except for Aronnson's remarkable results concerning classical solutions of this equation and examples of nonclassical "absolutely minimizing" functions, the theory did not advance significantly for over 25 years. Then R. Jensen was able to prove uniqueness for the Dirichlet problem when the equation is understood in the sense of viscosity solutions. Since Jensen's results, a few more keystones of the theory of the infinity-Laplace equation have been put in place and a number of significant extensions and applications have appeared. We will give a historical sketch of the mathematical developments as they concern the equation $\Delta_\infty u = 0$ itself, beginning with how it arose in Aronsson's work.

The claims listed below are just a chapter in the saga; the display is one simple way to highlight the special nature of Δ_∞ which we would like to share. Only some of this material will reappear in the lecture itself.

The Following are Equivalent!

Second order version:

$$\Delta_\infty u \geq 0 \text{ in } U \text{ in the viscosity sense, \quad aka, } u \text{ is "infinity subharmonic".}$$

First order version:

$$|Du(x)| \leq \max_{\{|w-x|=r\}} \frac{u(w)-u(x)}{r} \quad \text{for } \overline{B}_r(x) \subset\subset U$$

aka, "the gradient estimate".

Zero order version #1:

$$u(x) \leq u(y) + \left(\frac{\max_{\{|w-x|=r\}} u(w) - u(y)}{r} \right) |x-y| \quad \text{for } x \in B_r(y) \subset\subset U$$

aka, "comparison with cones from above".

Zero order version #2 (almost the same as above):

$$\max_{w \in \overline{B}_r(y)} u(w) - u(y) \leq \left(\max_{w \in \overline{B}_r(y)} u(w) - u(x) \right) \frac{r}{r-|x-y|}$$

for $x \in B_r(y) \subset\subset U$ aka, "the Harnack inequality".

Convexity version:

$$r \mapsto \max_{w \in \overline{B}_r(x)} u(w) \text{ is convex in } r, 0 \leq r < R = \operatorname{dist}(x, \partial U).$$

Variational Property:

If $V \subset\subset U$, $v \in C(\overline{V})$, $u \geq v$ in V and $u = v$ on ∂V, then

$$\| |Du| \|_{L^\infty(V)} \leq \| |Dv| \|_{L^\infty(V)}$$

aka, "u is sub-absolutely minimizing for $F(u) = \| |Du| \|_{L^\infty}$" (or some such).