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Field of Moduli of Rational Maps

Abstract If $\mathbb{K}$ is any subfield of $\mathbb{C}$, then we denote by $\mathrm{Rat}_d(\mathbb{K})$ the space of rational maps of degree d whose coefficients belong to $\mathbb{K}$. We set $\mathrm{Rat}_d = \mathrm{Rat}_d(\mathbb{C})$.

A complex rational map can be written in the form $R(z) = P(z)/Q(z)$, where $P(z), Q(z) \in \mathbb{C}[z]$ are relatively prime polynomials; in which case the degree of R is given by the maximum between the degrees of P and Q. If $P(z) = a_0 + a_1 z + \cdots + a_d z^d$ and $Q(z) = b_0 + b_1 z + \cdots + b_d z^d$, then the condition for R to have degree d is that either $a_d \neq 0$ or $b_d \neq 0$. So there is a natural injective map $\phi : \mathrm{Rat}_d \hookrightarrow \mathbb{P}^{2d+1}_{\mathbb{C}}$ defined as $\phi(R) = [a_0 : \cdots : a_d : b_0 : \cdots : b_d]$. In this case, as the condition for P and Q to be relatively prime is equivalent to have the resultant $\mathrm{Res}(P,Q) \neq 0$, the space Rat_d can be identified via ϕ with the Zariski open set $\mathbb{P}^{2d+1}_{\mathbb{C}} - X$, where X is the hypersurface defined by $\mathrm{Res}(P,Q) = 0$. In particular, Rat_d is a complex manifold of dimension $2d+1$. Notice that $\mathrm{Rat}_1 = \mathbb{M} = \mathrm{PGL}_2(\mathbb{C})$ is the group of Möbius transformations; a complex Lie group of dimension 3.

If $T \in \mathbb{M}$, and $R \in \mathrm{Rat}_d$, then $T \circ R \circ T^{-1} \in \mathrm{Rat}_d$. We say that R and S are equivalent rational maps (denoted this by the symbol $R \sim S$) if they belong to the same orbit under this action of $\mathbb{M}$. The quotient space $\mathrm{M}_d = \mathrm{Rat}_d/\mathbb{M}$ is the moduli space of rational maps of degree d. The space M_1 can be identified with the Riemann sphere with two cone points of order two, they correspond to the classes of $R(z) = z$ and $R(z) = -z$, respectively, and another special point corresponding to class of $R(z) = z+1$. If $d \geq 2$, then M_d has a natural structure of an affine geometric quotient [4] and the structure of a complex orbifold of dimension $2d-2$ (Milnor proved that $\mathrm{M}_2 \cong \mathbb{C}^2$ [2]). Explicit models for M_d seems not to be known for $d \geq 3$.

Let us denote by $\Gamma = \mathrm{Gal}(\mathbb{C})$ the group of field automorphisms of $\mathbb{C}$. If $R \in \mathrm{Rat}_d$ and $\sigma \in \Gamma$, then σ acts on R, by applying σ to the coefficients of R; we get in this way a rational map $R^\sigma \in \mathrm{Rat}_d$ [5]. In general, it may be that R^σ is not equivalent to R. Notice that if $R \sim S$ and $\sigma \in \Gamma$, then $R^\sigma \sim S^\sigma$, in particular, Γ induces an action on the moduli space M_d. The Γ-stabilizer of the class $[R] \in \mathrm{M}_d$ is given by the group $\Gamma_R := \{\sigma \in \Gamma : R^\sigma \sim R\}$; its fixed field $\mathcal{M}_R = \mathrm{Fix}(\Gamma_R) < \mathbb{C}$ is called the (absolute) field of moduli of R. Notice from the definition that if $R \sim S$, then $\Gamma_R = \Gamma_S$ and $\mathcal{M}_R = \mathcal{M}_S$. For instance, the quadratic polynomial $R_c(z) = z^2 + c$, where $c \in \mathbb{C}$, has field of moduli $\mathbb{Q}(c)$; which in this case is a field of definition. This comes from the fact that $R_c \sim R_d$ if and only if $c = d$.

A field of definition of $R \in \mathrm{Rat}_d$ is a subfield $\mathbb{K}$ of $\mathbb{C}$ so that there is some $S \in \mathrm{Rat}_d(\mathbb{K})$ with $S \sim R$. Every field of definition of R contains $\mathcal{M}_R$. In fact, if $\mathbb{K}$ is a field of definition of R, then (up to equivalence) we may assume that R is already defined over it. If $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{K})$, then $R^\sigma = R$; in particular $\sigma \in \Gamma_R$.

If $d \geq 2$ is even, then Silverman [3] proved that the field of moduli is a field of definition. He also proves that for polynomials maps this is true. In the same paper, if $d \geq 3$ is odd, then Silverman considered polynomials

$$R(z) = i\left(\frac{z-1}{z+1}\right)^d$$

and proved that they have field of moduli equal to $\mathbb{Q}$, but that they cannot be definable over it (they even cannot be definable over $\mathbb{R}$ since there is not a circle on the Riemann sphere $\widehat{\mathbb{C}}$ invariant under R). In these examples, the rational map is definable over a degree two extension over its field of moduli.

In this talk I present the following general fact.

Theorem. *Every rational map is definable over an extension of degree at most two of its field of moduli.*

We will also present necessary and sufficient conditions for a rational map to have a real field of moduli and also to be defined over the reals. Moreover, we provide a simple condition for a rational map to be definable over $\overline{\mathbb{Q}}$.

References

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