Analysis of an Ill-Posed Cauchy Problem for a Convection-Diffusion Equation

Z. Ranjbar
Linköping University,
SE-581 83 Linköping, Sweden
e-mail: zoran@math.liu.se

Abstract

We consider the following Cauchy problem for a convection-diffusion equation,

$$\begin{array}{ll} u_t = au_{xx} + 2bu_x + cu, & 0 \le x \le 1, \quad 0 \le t \le 1, \quad a > 0, \\ u(x,0) = 0, & 0 \le x \le 1, \\ u_x(0,t) = 0, & 0 \le t \le 1, \\ u(0,t) = g(t), & 0 \le t \le 1. \end{array} \tag{1}$$

From the data at the left boundary we want to determine the solution at the right boundary $u(1,t) = f(t)$, where u is concentration of certain species, a is diffusion constant, b is flow velocity and c is reaction coefficient [3]. The inverse problem (1) is studied in [5].

Non-characteristic Cauchy problems for parabolic equations are well-known to be severely ill-posed in the sense that a small perturbation in the data g may cause a large error in the solution f, see e.g. the survey [1] and the references therein. Stability can be restored by imposing a bound on the solution. Some stability result for (1) are given in [4, 2].

In this paper we study how the convection term influences the degree of ill-posedness. We compute numerically the singular values and corresponding singular functions (vectors) of the equivalent Volterra integral equation of the first kind. We show that the coefficient in the convection term determines the rate of decay of the singular values, and thus it influences the degree of ill-posedness. We also discuss how the stability depends on the convection term. Some numerical examples are given.

This is joint work with L. Eldén.

References

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